Based on readings of Prof. Kerry E. Back on the IAS in Vienna, October 2001.

Advanced topics in continuous time finance

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November 2001

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1 Introduction

1.1 Martingale

$(M_t)_{0\leq t\leq\infty}$ is a martingale if:

$$M_s = \mathbb{E}\left[M_t|\mathcal{F}_s\right] \quad \forall s\leq t$$

$(M_t)_{0\leq t\leq T}$ is a martingale if:

$$M_t = \mathbb{E}\left[M_T|\mathcal{F}_t\right] \quad \forall t$$

1.2 Change of measure

Let $x>0$ a.s. Define

$$Q\left(A\right) = \frac{1}{\mathbb{E}\left[x\right]}\cdot\mathbb{E}\left[x\cdot I_A\right] \quad \text{for all events } A\in\mathcal{E}$$

Then Q is a probability measure equivalent to P meaning $Q\left(A\right)=0$ iff $P\left(A\right)=0$.
Write $\frac{dQ}{dP}=\frac{X}{\mathbb{E}[X]}$ Radon-Nikodyn derivative of Q with respect to P.

$$Q\left(A\right) = \mathbb{E}^Q\left[I_A\right] = \int_A dQ = \frac{1}{\mathbb{E}\left(x\right)}\cdot\mathbb{E}\left[x\cdot I_A\right] = \int_A \frac{x}{\mathbb{E}\left[x\right]}dP$$

1.3 Security Price Model

$Z\left(t\right)=$asset price

$D\left(t\right)=$cumulative dividend process (assume: $D\left(t\right)$ is increasing)

Example: Proportional dividend

$$\begin{aligned} dD\left(t\right) &= \delta\left(t\right)\cdot Z\left(t\right)dt \\ D\left(t\right) &= \int_0^t \delta\left(u\right)\cdot Z\left(u\right)du \end{aligned}$$

$S(t)$ =re-invested asset price

Defintion: $S(t) = Z(t) \cdot e^{\int_0^t \delta(u)du} = Z(t) \cdot e^{\int_0^t \frac{1}{Z(u)} dD(u)}$

Write: $\theta(t) = e^{\int_0^t \frac{1}{Z(u)} dD(u)}$

1.4 Proportional dividends

$$\begin{aligned} dD(t) &= \delta(t) \cdot Z(t)\, dt \\ \theta(t) &= e^{\int_0^t \delta(u)du} \\ \theta'(t) &= \theta(t) \cdot \delta(t) \\ d\theta &= \theta(t) \cdot \delta(t)\, dt \end{aligned}$$

S =value process of a self financing trading strategy.

Example: money market account

Suppose a risk free interest rate r

Meaning there exists a security with price process $S(t) = e^{rt}$

Currency in a bank

$$\begin{aligned} Z(t) &= 1 \\ S(t) &= Z(t) \cdot e^{\int_0^t r du} \\ &= e^{rt} \end{aligned}$$

1.5 Stock price density (Pricing kernel)

Absence of arbitrage implies: there exists a stochastic process ρ such that for all T all reinvested asset processes $S_1,\, S_2,\, \forall t \leq u$ and event $A \in \mathcal{F}_t$:

$$\mathbb{E}\left[\rho(T) \cdot S_1(T) \cdot I_A \cdot \left\{\frac{S_2(u)}{S_1(u)} - \frac{S_2(t)}{S_1(t)}\right\}\right] = 0 \tag{1.1}$$

More fundamentally: for any self financing trading strategy θ with value process V_θ it is valid:

$$V_\theta(0) = \mathbb{E}\left[\rho(T) \cdot V_\theta(T)\right] \tag{1.2}$$

ρ =Arrow Debren state prices

Proof of 1.1 from 1.2

We assume no dividends. Consider the following self financing strategy:

- In event A at time t buy one unit of security 2 and sell $\frac{S_2(t)}{S_1(t)}$ units of security 1.
- At time u sell the unit of security 2 and buy $\frac{S_2(u)}{S_1(u)}$ units of security 1.
- Hold until T:

$$\begin{aligned} V_\theta(T) &= S_1(T) \cdot \left\{ \frac{S_2(u)}{S_1(u)} - \frac{S_2(t)}{S_1(t)} \right\} \cdot I_A \\ V_\theta(0) &= 0 \end{aligned}$$

This implies 1.1 (Arbitrage!).

Make the change of measure

$$\begin{aligned} \frac{dQ}{dP} &= \frac{\rho(T) \cdot S_1(T)}{S_1(0)} \\ \mathbb{E}\left[\rho(T) \cdot S_1(T)\right] &= S_1(0) \\ \mathbb{E}^Q\left[I_A \cdot \left\{ \frac{S_2(u)}{S_1(u)} - \frac{S_2(t)}{S_1(t)} \right\} \right] &= 0 \\ &\Updownarrow \\ \mathbb{E}^Q\left[\frac{S_2(u)}{S_1(u)} \cdot I_A \right] &= \mathbb{E}^Q\left[\frac{S_2(t)}{S_1(t)} \cdot I_A \right] \qquad \forall A \in \mathcal{F}_t, \forall t \le u \\ \frac{S_2(t)}{S_1(t)} &= \mathbb{E}^Q\left[\left. \frac{S_2(u)}{S_1(u)} \right| \mathcal{F}_t \right] \qquad \forall t \le u \end{aligned}$$

So it follows that

$$\left(\frac{S_2(t)}{S_1(t)} \right)_{0 \le t \le T}$$

is a Q-martingale.

Example

$$\begin{aligned} S_1(t) &= e^{rt} \\ e^{-rt} \cdot S(t) &= \text{Q} - \text{martingale } \forall S \\ e^{-rt} \cdot S(t) &= \mathbb{E}^Q\left[e^{-rT} \cdot S(T) \middle| \mathcal{F}_t \right] \end{aligned}$$

We say that we are “using security 1 as the *numeraire*”.

2 Forward contracts

Lets define $F(t)$ as the forward price at time t to receive X at time T. If you go long one forward contract at time t in event $A \in \mathcal{F}_t$ you profit $[X - F(t)] \cdot I_A$:

$$0 = \mathbb{E}\left[\rho(T) \cdot [X - F(t)] \cdot I_A\right]$$

The goal is that

$$\left(\frac{S_2(t)}{S_1(t)}\right)_{0 \le t \le T}$$

is a Q-martingale.

Take S_1 to be a discount bond (zero bond) maturing at time T ($S_1(t) = e^{-r(T-t)}$). If there exists a constant risk free rate then it follows:

$$\begin{aligned}
\mathbb{E}\left[\rho(T) \cdot F(t) \cdot I_A\right] &= \mathbb{E}\left[\rho(T) \cdot X \cdot I_A\right] \\
S_1(0) \cdot \mathbb{E}\left[\rho(T) \cdot \frac{S_1(T)}{S_1(0)} \cdot F(t) \cdot I_A\right] &= S_1(0) \cdot \mathbb{E}\left[\rho(T) \cdot \frac{S_1(T)}{S_1(0)} \cdot X \cdot I_A\right]
\end{aligned}$$

Now write:

$$\begin{aligned}
\frac{dQ}{dP} &= \rho(T) \cdot \frac{S_1(T)}{S_1(0)} \\
\mathbb{E}^Q\left[F(t) \cdot I_A\right] &= \mathbb{E}^Q\left[X \cdot I_A\right] \\
F(t) &= \mathbb{E}^Q\left[X \mid \mathcal{F}_t\right]
\end{aligned}$$

We see that F is a Q-martingale. Changing the measure using a discount bond as numeraire is called the *forward measure*.

If there is a constant risk free rate r then $\frac{S_1(T)}{S_1(0)} = e^{rT}$. Using a discount bond as numeraire gives you some Q as using money market account.

More generally if the risk free rate is deterministic then the price of a discount bond equals

$$\begin{aligned}
S_1(t) &= e^{-\int_t^T r(u)du} \\
\frac{S_1(T)}{S_1(0)} &= e^{\int_0^T r(u)du}
\end{aligned}$$

Example: Suppose you go long one forward at time 0 and buy $F(0)$ discount bonds. What is the value at time t?

At time T you receive $F(t) - F(0)$. The value at t is equal to $[F(t) - F(0)] \cdot S_1(t)$ where $S_1(t)$ is the price of a discount bond maturing at time T. A portfolio including $F(0)$ discount bonds is worth $F(t) \cdot S_1(t)$.

Now check if this is valid at time T:

$$\begin{aligned} V(T) &= X \\ F(T) \cdot \underbrace{S_1(T)}_{1} = F(T) &= X \end{aligned}$$

Making definitions more understandable

$X = \mathbb{E}[y|\mathcal{F}_t]$

Definition: $\mathbb{E}[x \cdot I_A] = \mathbb{E}[y \cdot I_A]$

Write $\varepsilon = y - x$ so that $y = x + \varepsilon$

Now take x to be $\mathcal{F}_t-$measurable

$$\begin{aligned} \mathbb{E}[y|\mathcal{F}_t] &= x + \mathbb{E}[\varepsilon|\mathcal{F}_t] \\ &= x \qquad \text{iff } \mathbb{E}[\varepsilon|\mathcal{F}_t] = 0 \end{aligned}$$

Definition:

$$\begin{aligned} \mathbb{E}[y \cdot I_A] &= \mathbb{E}[(x+\varepsilon) \cdot I_A] \\ &= \mathbb{E}[x \cdot I_A] + \mathbb{E}[\varepsilon \cdot I_A] \\ &= \mathbb{E}[x \cdot I_A] \qquad \text{iff } \mathbb{E}[\varepsilon \cdot I_A] = 0 \quad \forall A \in \mathcal{F}_t \end{aligned}$$

We see that

$$\mathbb{E}[\varepsilon|\mathcal{F}_t] = 0 \quad \text{iff} \quad \mathbb{E}[\varepsilon \cdot I_A] = 0 \quad \forall A \in \mathcal{F}_t$$

Define

$$\begin{aligned} \mathbb{E}[\varepsilon|A] &= \frac{\int_A \varepsilon dP}{P(A)} \\ &= \frac{\mathbb{E}[I_A \cdot \varepsilon]}{P(A)} \\ P(B|A) &= \frac{P(B \cap A)}{P(A)} \\ &= \frac{\mathbb{E}[I_A \cdot I_B]}{P(A)} \\ &= \frac{\int_A I_B(w)\, dP(w)}{P(A)} \\ \mathbb{E}[\varepsilon \cdot I_A] &= 0 \quad \forall A \in \mathcal{F}_t \end{aligned}$$

We see that defining $\mathbb{E}\left[\varepsilon\right|\mathcal{F}_t] = 0$ means that $\mathbb{E}\left[\varepsilon\right| A] = 0 \quad \forall A \in \mathcal{F}_t$

2.1 Forward on a dividend paying security

We assume that the dividends are deterministic and proportional.

$$\begin{aligned} dD(t) &= \delta(t) \cdot Z(t)\, dt \qquad \delta \text{ is non} - \text{random} \\ S(t) &= Z(t) \cdot e^{\int_0^t \delta(u) du} \end{aligned}$$

The forward contract has a value of $Z(T) - F(0)$ at time T. A Portfolio of a long forward and long $F(0)$ discount bonds has a value of $F(0) \cdot S_1(t)$ or $F(0) \cdot S_1(T) = Z(T)$ at time T (S_1 is the price of the discount bond).

An equivalent portfolio would be buying $e^{-\int_0^T \delta(u) du}$ shares of the security at time 0. At time T we then have one share with value $Z(T)$. We see that

$$\begin{aligned} e^{-\int_0^T \delta(u) du} \cdot Z(0) &= F(0) \cdot S_1(0) \\ F(0) &= \underbrace{\frac{1}{S_1(0)}}_{e^{\int_t^T r(u) du}} \cdot e^{-\int_0^T \delta(u) du} \cdot Z(0) \\ &= e^{\int_0^T (r(u) - \delta(u)) du} \cdot Z(0) \end{aligned}$$

if r is deterministic.

3 Stock options

The value of an european call with strike price K and maturity T at time 0 is equal to

$$\mathbb{E}\left[\rho(T)\left[Z(T)-K\right]^{+}\right]=\underbrace{\mathbb{E}\left[\rho(T)\cdot Z(T)\right]\cdot I_{\{Z(T)>K\}}}_{\clubsuit}+\underbrace{\mathbb{E}\left[\rho(T)\cdot K\cdot I_{\{Z(T)>K\}}\right]}_{\diamondsuit}$$

$\diamondsuit$: Use S_1 as price of a discount bond maturing at T

$$\begin{aligned}\mathbb{E}\left[\rho(T)\cdot K\cdot I_{\{Z(T)>K\}}\right] &= S_1(0)\cdot\mathbb{E}\left[\rho(T)\cdot\frac{S_1(T)}{S_1(0)}\cdot K\cdot I_{\{Z(T)>K\}}\right]\\ &= K\cdot S_1(0)\cdot\mathbb{E}^{Q}\left[I_{\{Z(T)>K\}}\right]\\ &= K\cdot S_1(0)\cdot Q\left(Z(T)>K\right)\end{aligned}$$

$\clubsuit$: $Z(T)$ equals the value of a portfolio of one forward on the security long and $F(0)$ discount bonds long. The value of the portfolio at time 0 is $F(0)\cdot S_1(0)$:

$$\begin{aligned}\mathbb{E}\left[\rho(T)\cdot Z(T)\cdot I_{\{Z(T)>K\}}\right] &= F(0)\cdot S_1(0)\cdot\mathbb{E}\left[\rho(T)\cdot\frac{Z(T)}{F(0)\cdot S_1(0)}\cdot I_{\{Z(T)>K\}}\right]\\ &= F(0)\cdot S_1(0)\cdot\mathbb{E}^{Q^*}\left[I_{\{Z(T)>K\}}\right]\\ &= F(0)\cdot S_1(0)\cdot Q^*\left(Z(T)>K\right)\end{aligned}$$

Specials case: deterministic proportional dividends

$$\begin{aligned}F(0)\cdot S_1(0) &= e^{-\int_0^T\delta(u)du}\cdot Z(0)\\ \frac{Z(T)}{F(0)\,S_1(0)} &= \frac{e^{-\int_0^T\delta(u)du}\cdot Z(T)}{Z(0)}\\ &= \frac{S(T)}{S(0)}\end{aligned}$$

Q^* is the measure obtained by using the reinvested asset price as numeraire. In particular, if $\delta=0$, then Q^* is obtained by using the stock price as numeraire. To really calculate the probabilities one has to be specific on the dynamics of S_1 and S_2; here comes the geometric brownian motion in.

4 International Finance

Lets define $X(t)$ as the exchange rate, i.e. the units of domestic currency per units of foreign currency. Let S^f be a reinvested asset price in foreing currency. Then $X(t) \cdot S^f(t)$ equals the reinvested asset price in domestic currency. If S_1 is a domestic asset then $\frac{X \cdot S^f}{S_1}$ is a Q-martingale, using S_1 as numeraire.

$$\mathbb{E}\left[\rho(T) \cdot S_1(T) \cdot I_A \cdot \left\{\frac{X(u) \cdot S^f(u)}{S_1(u)} - \frac{X(t) \cdot S^f(t)}{S_1(t)}\right\}\right] = 0 \qquad \forall A \in \mathcal{F}_t,\, \forall t \leq u \leq T \tag{4.1}$$

Take $S_1(t) = X(t) \cdot S_1^f(t)$ for some foreign currency S_1^f.

$$\mathbb{E}\left[\rho(T) \cdot X(t) \cdot S_1^f(t) \cdot I_A \cdot \left\{\frac{S^f(u)}{S_1^f(u)} - \frac{S^f(t)}{S_1^f(t)}\right\}\right] = 0$$

Now define ρ^f as

$$\rho^f(t) = \frac{\rho(t) \cdot X(t)}{X(0)} \tag{4.2}$$

It follows:

$$X(0) \cdot \mathbb{E}\left[\rho^f(T) \cdot S_1^f(t) \cdot I_A \cdot \left\{\frac{S^f(u)}{S_1^f(u)} - \frac{S^f(t)}{S_1^f(t)}\right\}\right] = 0$$

This looks like in chapter 2 on forwards.

For all foreign reinvested assets prices S_2^f the expression $\frac{S_2^f(t)}{S_1^f(t)}$ is a Q^f-martingale where

$$\frac{dQ^f}{dP} = \frac{\rho^f(T) \cdot S_1^f(T)}{S_1^f(0)}$$

P is the same all over the world!

Price of any foreign currency denominated claim y^f at time T

The value at time $t = 0$ is equal to

$$\mathbb{E}\left[\rho(T) \cdot X(T) \cdot y^f(T)\right] = X(0) \cdot \mathbb{E}\left[\frac{\rho(T) \cdot X(T)}{X(0)} \cdot y^f(T)\right]$$

Now plug in formula 4.2 and we get

$$X(0) \cdot \mathbb{E}\left[\rho^f(T) \cdot y^f(T)\right]$$

Foreign currency value at $t = 0$ is

$$\mathbb{E}\left[\rho^f(T) \cdot y^f(T)\right]$$

Example: Think of a foreign stock option with strike price set in foreign currency. No dividends are paid and the risk free rate is assumed to be constant. What is the value at $t = 0$ in domestic currency?

The simplest approach is to calculate (compare with chapter 3)

$$X(0) \cdot \mathbb{E}\left[\rho^f(T)\left[S^f(T) - K^f\right]^+\right] =$$
$$= X(0) \cdot \left\{ \underbrace{F(0) \cdot S_1(0)}_{S(0)} \cdot Q^*(S(T) > K) - K \cdot \underbrace{S_1(0)}_{e^{-rt}} \cdot Q(S(T) > K) \right\}$$

Q^* is using the stock as numeraire; Q is using the money market account as numeraire.

Example: Same as above, but now the strike price is in domestic currency.

$$\mathbb{E}\left[\rho(T) \cdot \left[X(T) \cdot S^f(T) - K\right]^+\right] =$$
$$= \mathbb{E}\left[\rho(T) \cdot X(T) \cdot S^f(T) \cdot I_{\left\{X(T) \cdot S^f(T) > K\right\}}\right] - \mathbb{E}\left[\rho(T) \cdot K \cdot I_{\left\{X(T) \cdot S^f(T) > K\right\}}\right]$$

The first part of the equation uses the foreign stock in foreign currency as numeraire; the second part uses the domestic discount bond as numeraire.

5 Definitions

5.1 Variations

For a function $f : \mathbb{R} \to \mathbb{R}$ the *total variation* of f is defined as

$$\sum_i |\Delta_i f|$$

The *quadratic variation* of f is defined as

$$\sum_i |\Delta_i f|^2$$

If the total variation of f is finite, then the quadratic variation is equal to zero ("f has finite variation"). Suppose f is continously differentiable, then

$$f(t) = f(0) + \int_0^t f\prime(x)\,dx$$

and f has finite variations. We write $df = f\prime(x)\,dx$.

Suppose M is a continous martingale. Then either

1. with probability one the paths of M

$$t \to M(t, \omega) \qquad \omega \in \Omega$$

   have finite variations
   OR

2. M is constant a.s.

Definition: We write the quadratic variation of M over $[0, t]$ as

$$\langle M, M \rangle (t)$$

Then $M^2 - \langle M, M \rangle$ is a martingale.

Definition: If M is a continous martingale then M is a Brownian Motion iff

$$\langle M, M\rangle (t) = t$$

We write $d\langle M, M\rangle = dt$ or $(dM)^2 = dt$

Definition: For two functions $f, g : \mathbb{R} \to \mathbb{R}$ we define

$$\langle f, g\rangle = lim \sum_i \Delta_i f \cdot \Delta_i g$$

If M and N are continous martingale then

$$MN - \langle M, N\rangle$$

is a martingale and $\langle M, N\rangle$ is called the *covariance process* of M and N.

5.2 Ito's Lemma

Let $dX = \mu(t)\, dt + \sigma(t)\, dW$ with W being a Wiener process. If $f = C^2$ (i.e. it is a smooth, twice differentiable function) then it follows

$$\begin{aligned}
f(X_t) &= f(X_0) + \int_0^t f\prime(X_u)\, dX_u + \frac{1}{2} \cdot \int_0^t f\prime\prime(X_u)\, d\langle X, X\rangle (u) \\
\langle X, X\rangle (t) &= \int_0^t \sigma(u)^2\, d\langle W, W\rangle \\
&= \int_0^t \sigma(u)^2\, du \\
d\langle X, X\rangle (t) &= \sigma(t)^2\, dt \\
df &= f\prime(x) \cdot \{\mu dt + \sigma dw\} + \frac{1}{2} \cdot f\prime\prime(x)\, \sigma^2 dt
\end{aligned}$$

Now let two processes X and Y be defined as

$$\begin{aligned}
dX &= \mu_X dt + \sigma_X dW_1 \\
dY &= \mu_Y dt + \sigma_Y dW_2
\end{aligned}$$

If f is a smooth function it follows

$$\begin{aligned}
df(X(t), Y(t)) &= \frac{\delta f}{\delta X} dX + \frac{\delta f}{\delta Y} dY + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta X^2} d\langle X, X\rangle + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta Y^2} d\langle Y, Y\rangle + \frac{\delta^2 f}{\delta X \delta Y} d\langle X, Y\rangle \\
d\langle X, X\rangle &= \sigma_X^2 dt \\
d\langle Y, Y\rangle &= \sigma_Y^2 dt \\
d\langle X, Y\rangle &= \sigma_X \cdot \sigma_Y\, d\langle W_1, W_2\rangle \\
d\langle W_1, W_2\rangle (t) &= \rho(t)\, dt \quad \text{for some } |\rho| < 1
\end{aligned}$$

5.3 Girsonov's Theorem

Let W be a Brownian motion on $\{\Omega, \mathcal{F}, P\}$ and Q be a probability measure equivalent to P. Now define $\xi = \frac{dQ}{dP}$ and set $\xi(t) = \mathbb{E}[\xi | \mathcal{F}_t]$. If we assume that

$$d\xi = -\lambda(t) \cdot \xi(t)\, dW(t) \quad \text{for some } \lambda$$

then

$$W^*(t) \hat{=} W(t) + \int_0^t \lambda(u)\, du$$

is a Brownian motion under Q. Further

$$\begin{aligned} dW^* &= dW + \lambda dt \\ \langle W^*, W^* \rangle &= \langle W, W \rangle \\ &= t \end{aligned}$$

5.4 Martingale representation theorem

Let the filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ be generated by the Brownian motion W. If M is a martingale, then there exists a Φ such that

$$\begin{aligned} dM &= \Phi dW \\ M(t) &= M(0) + \int_0^t \Phi_1(s)\, dW_1(s) + \int_0^t \Phi_2(s)\, dW_2(s) + \ldots \end{aligned}$$

We know that $d\xi = \Phi dW$ and we write $\lambda = -\frac{\Phi}{\xi}$.

Equivalent statement

The following statement is a Q-Brownian Motion:

$$dW^* = dW - \frac{1}{\xi} d\langle \xi, W \rangle$$

We have

$$d\langle \xi, W \rangle = -\lambda \cdot \xi d\langle W, W \rangle$$

so that

$$-\frac{1}{\xi} d\langle \xi, W \rangle = \lambda dt$$

A Brownian motion W under P with up/down probability equal to $\frac{1}{2}$:

$$
\begin{array}{ccccc}
 & & & & 2 \cdot \sqrt{dt} \\
 & & & \nearrow & \\
 & & \sqrt{dt} & & \\
 & \nearrow & & \searrow & \\
\text{Time } 0 & & & & 0 \\
 & \searrow & & \nearrow & \\
 & & \sqrt{dt} & & \\
 & & & \searrow & \\
 & & & & 2 \cdot \sqrt{dt}
\end{array}
$$

Now we look at W^* with drift:

$$
\begin{array}{ccccc}
 & & & & 2 \cdot \sqrt{dt} + 2 \cdot \lambda dt \\
 & & & \nearrow & \\
 & & \sqrt{dt} + \lambda dt & & \\
 & \nearrow & & \searrow & \\
\text{Time } 0 & & & & 2 \cdot \lambda dt \\
 & \searrow & & \nearrow & \\
 & & \sqrt{dt} + \lambda dt & & \\
 & & & \searrow & \\
 & & & & 2 \cdot \sqrt{dt} + 2 \cdot \lambda dt
\end{array}
$$

To make it a martingale the up probability must be

$$\frac{1 - \lambda\sqrt{dt}}{2} = \frac{1 - \lambda dW}{2}$$

and the down probability must be[1]

$$\frac{1 + \lambda\sqrt{dt}}{2} = \frac{1 - \lambda dW}{2}$$

$$\frac{Q}{P} = 1 - \lambda dW$$

First time step: $\frac{Q_1}{P_1} = 1 - \lambda dW_1$

Second time step: $\frac{Q_2}{P_2} = \underbrace{(1 - \lambda dW_1)}_{\frac{Q_1}{P_1}} \cdot (1 - \lambda dW_2)$

General: $\frac{\Delta \frac{Q}{P}}{\frac{Q}{P}} = \frac{\frac{Q_2}{P_2} - \frac{Q_1}{P_1}}{\frac{Q_1}{P_1}} = -\lambda dW$

[1]Watch the sign!

From the theorem we see

$$\begin{aligned} \frac{d\xi}{\xi} &= -\lambda dW \\ \frac{Q(A)}{P(A)} &= \mathbb{E}\left[\xi \middle| A\right] \end{aligned}$$

6 Valuation of stock options

We assume a constant risk free rate r and no dividends. The value at time $t = 0$ is equal to

$$\underbrace{S_0 \cdot Q^* \left(S\left(T\right) > K\right)}_{\clubsuit} - \underbrace{e^{-rT} \cdot K \cdot Q\left(S\left(T\right) > K\right)}_{\diamondsuit}$$

where Q^* uses the stock as numeraire and Q uses the money market as numeraire. We further assume that $\frac{dS}{S} = \mu dt + \sigma dW$ for constant μ and σ.

$\diamondsuit$: Under Q $\frac{S}{e^{rt}}$ is a martingale (compare with chapter 5.2):

$$\begin{aligned} dX &= \mu_X dt + \sigma_X dW_1 \\ dY &= \mu_Y dt + \sigma_Y dW_2 \\ F\left(X, Y\right) &= \frac{X}{Y} \quad \left(= \frac{S}{e^{rt}}\right) \end{aligned}$$

$$\begin{array}{lll} \frac{\delta F}{\delta X} = \frac{1}{Y} & \frac{\delta F}{\delta Y} = -\frac{X}{Y^2} & \\ \frac{\delta^2 F}{\delta X^2} = 0 & \frac{\delta^2 F}{\delta Y^2} = \frac{2 \cdot X}{Y^3} & \frac{\delta F}{\delta X \delta Y} = -\frac{1}{Y^2} \end{array}$$

$$\begin{aligned} d\left(\frac{X}{Y}\right) &= \frac{dX}{Y} - \frac{X dY}{Y^2} + \frac{1}{2} \cdot 2 \cdot \frac{X}{Y^3} d\left\langle Y, Y\right\rangle - \frac{1}{Y^2} d\left\langle X, Y\right\rangle \\ \frac{d\left(\frac{X}{Y}\right)}{\frac{X}{Y}} &= \frac{dX}{X} - \frac{dY}{Y} + \frac{d\left\langle Y, Y\right\rangle}{Y^2} - \frac{d\left\langle X, Y\right\rangle}{X \cdot Y} \\ \frac{d\left(\frac{S}{e^{rt}}\right)}{\frac{S}{e^{rt}}} &= \frac{dS}{S} - r dt \\ &= \left(y - r\right) dt + \sigma dW_1 \end{aligned}$$

Now set $\xi = \frac{dQ}{dP}$. By the martingale representation theorem

$$\frac{d\xi}{\xi} = -\lambda dW \qquad \text{for some } \lambda$$

By Girsonov's theorem we know that

$$dW^* = dW + \lambda dt$$

is a $Q-$Brownian motion.

$$\begin{aligned}\frac{d\left(\frac{S}{e^{rt}}\right)}{\frac{S}{e^{rt}}} &= (\mu - r)\,dt + \sigma \cdot (dW^{*} - \lambda dt) \\ &= (\mu - r - \sigma \cdot \lambda)\,dt + \sigma dW^{*}\end{aligned}$$

The difference of two martingales is again a martingale. So

$$\mu - r - \sigma \cdot \lambda = 0$$

or if we rewrite the term

$$\lambda = \frac{\mu - r}{\sigma}$$

This term is called the *price of risk*.

We know that

$$\begin{aligned}\frac{dS}{S} &= \mu dt + \sigma dW \\ &= \mu dt + \sigma\,(dW^{*} + \lambda dt) \\ &= r dt + \sigma dW^{*}\end{aligned}$$

So we can calculate

$$\begin{aligned}S(t) &= S(0) \cdot e^{rt - \frac{1}{2}\cdot\sigma^{2}\cdot t + \sigma W(t)} \\ \log S(t) &= \log S(0) + \left(r - \frac{1}{2}\cdot\sigma^{2}\right)\cdot t + \sigma W(t) \\ Q\left(S(T) > K\right) &= Q\left(\log S(T) > \log K\right) \\ &= Q\left(\sigma W(T) > \log K - \log S(0) - \left(r - \frac{1}{2}\cdot\sigma^{2}\right)\cdot T\right) \\ &= Q\left(-\frac{W(T)}{\sqrt{T}} < d_2\right) \\ &= N\left[d_2\right] \\ d_2 &= \frac{\log S(0) - \log K + \left(r - \frac{1}{2}\cdot\sigma^{2}\cdot T\right)}{\sigma\cdot\sqrt{T}}\end{aligned}$$

Q is unique!

♣: Now calculate $S(0) \cdot Q^{*}\left(S(T) > K\right)$ with Q^{*} using the stock as numeraire.

We know that $\frac{e^{rt}}{S(t)}$ is a Q^*-martingale and that for some λ the term $dW^* = dW + \lambda dt$ defines a Q^* Brownian motion. Further we know

$$\begin{aligned} dS &= \mu \cdot S dt + \sigma \cdot S dW \\ d\langle S, S\rangle &= \sigma^2 \cdot S^2 dt \end{aligned}$$

Using Ito we calculate

$$\begin{aligned} \frac{d\left(\frac{e^{rt}}{S(t)}\right)}{\frac{e^{rt}}{S(t)}} &= r dt - \frac{dS}{S} + \frac{d\langle S, S\rangle}{S^2} \\ &= r dt - \mu dt - \sigma dW + \sigma^2 dt \\ &= \left(r - \mu + \sigma^2\right) dt - \sigma\left(dW^* - \lambda dt\right) \\ &= \left(r - \mu + \sigma^2 + \sigma \cdot \lambda\right) dt - \sigma dW^* \end{aligned}$$

We conclude

$$\begin{aligned} r - \mu + \sigma^2 + \sigma \cdot \lambda &= 0 \\ \frac{dS}{S} &= \mu dt + \sigma \cdot \left(dW^* - \lambda dt\right) \\ &= \left(\mu + r - \mu + \sigma^2\right) dt + \sigma dW^* \\ &= \left(r + \sigma^2\right) dt + \sigma dW^* \\ \log S(t) &= \log S(0) + \left(r + \sigma^2 - \frac{1}{2} \cdot \sigma^2\right) \cdot t + \sigma W^*(t) \\ Q^*\left(S(T) > K\right) &= Q^*\left(-\frac{W^*(T)}{\sqrt{T}} < d_1\right) \\ &= N\left[d_1\right] \\ d_1 &= \frac{\log S(0) - \log K + \left(r + \frac{1}{2} \cdot \sigma^2\right) T}{\sigma \cdot \sqrt{T}} \end{aligned}$$

7 Stochastic parameters

7.1 Stochastic interest rates

We know that $\frac{dS}{S} = \mu dt + \sigma(t)\, dW$ and that the money market account is equal to $e^{\int_0^t r(u)du}$. Discount bond maturing at T with bond price B and dynamics $dB = \mu_B dt + \sigma_B dW$. The forward measure uses the discount bond as numeraire.

The second term in Black-Scholes:

$$B(0) \cdot K \cdot Q(S(T) > K) = \mathbb{E}\left[\rho(T) \cdot K \cdot I_{\{S(T)>K\}}\right]$$

We calculate

$$\begin{aligned}
\frac{e^{\int_0^t r(u)du}}{B(t)} &= Q - \text{martingale} \\
\frac{d\left(\frac{e^{\int_0^t r}}{B}\right)}{\frac{e^{\int_0^t r}}{B}} &= rdt - \frac{dB}{B} + \frac{d\langle B, B\rangle}{B^2} \\
&= \left(r - \mu_B + \sigma_B^2\right) dt - \sigma_B dW
\end{aligned}$$

Using $dW^* = dW + \lambda dt$ it follows

$$r - \mu_B + \sigma_B^2 + \sigma_B \cdot \lambda = 0$$

Further we calculate

$$\begin{aligned}
\frac{dS}{S} &= \mu dt + \sigma dW \\
&= \mu dt + \sigma \cdot (dW^* - \lambda dt) \\
&= \left(\mu - \frac{\sigma}{\sigma_B} \cdot \left[\mu_B - r - \sigma_B^2\right]\right) dt + \sigma dW^* \\
\log S(t) &= \log S(0) + \int_0^t \left(\mu - \frac{\sigma}{\sigma_B} \cdot \left[\mu_B - r - \sigma_B^2\right] - \frac{1}{2} \cdot \sigma^2\right) du + \int_0^t \sigma dW^* \\
&= Q^* \left(\int_0^T \left\{\mu - \frac{\sigma \cdot \mu_B}{\sigma_B} + \sigma \cdot \sigma_B - \frac{1}{2} \cdot \sigma^2\right\} du + \int_0^T \frac{\sigma}{\sigma_B} \cdot r du + \int_0^T \sigma dW > logK\right)
\end{aligned}$$

The distribution of

$$\int_0^T \sigma(t)\, dW^*(t)$$

is given by

$$N\left[0, \int_0^T \sigma(t)^2\, dt\right]$$

In Vasicek

$$\int_0^T \frac{\sigma}{\sigma_B} \cdot r dt \sim N$$

7.2 Stochastic volatility

We are using Heston's model with a constant risk free rate and no dividends.

$$\frac{dS}{S} = \mu dt + \sqrt{v} dW_1 \qquad v \geq 0$$

The distribution of v follows a mean-reverting CIR square root model

$$dv = \kappa \cdot (\theta - v)\, dt + \sigma \cdot \sqrt{v} dW_2$$

whereas θ equals to the long run mean.

Value of a call option

$$\underbrace{S(0) \cdot Q^*(S(T) > K)}_{\clubsuit} - \underbrace{e^{-rT} \cdot K \cdot Q(S(T) > K)}_{\diamondsuit}$$

Q^* is using the stock as numeraire, Q is using the money market account as numeraire.

$\diamondsuit$: The term $e^{-rt} \cdot S$ is a $Q-$martingale

$$\begin{aligned}
\frac{d\left(e^{-rt} \cdot S\right)}{e^{-rt} \cdot S} &= \frac{dS}{S} - r dt \\
&= (\mu - r)\, dt + \sqrt{v} dW_1 \\
\xi &= \frac{dQ}{dP} \\
\frac{d\xi}{\xi} &= -\lambda_1 dW_1 - \lambda_2 dW_2 \\
dW_1^* &= dW_1 - \frac{1}{\xi} d\langle \xi, W_1 \rangle \\
&= dW_1 + \lambda_1 d\langle W_1, W_1 \rangle + \lambda_2 d\langle W_1, W_2 \rangle \\
&= dW_1 + \lambda_1 dt + \lambda_2 \cdot \rho dt \\
&= (\mu - r)\, dt + \sqrt{v}\, (dW_1^* - \lambda_1 dt - \lambda_2 \rho dt) \\
\rho dt &= d\langle W_1, W_2 \rangle = \text{const.}
\end{aligned}$$

It must be

$$\mu - r - \lambda_1 \cdot \sqrt{v} - \lambda_2 \cdot \rho \cdot \sqrt{v} = 0$$

We have one equation but two unknowns (λ_1 and λ_2); we have an incomplete market and can't price via arbitrage. So we do *equilibrium pricing*, i.e. we are making some assumptions.

$$\begin{aligned}
\frac{dS}{S} &= \mu dt + \sqrt{v} dW_1 \\
&= \mu dt + \sqrt{v} \cdot \left(dW_1^* - \lambda_1 dt - \lambda_2 \cdot \rho dt\right) \\
&= r dt + \sqrt{v} dW_1^* \\
dv &= \kappa \cdot (\theta - v)\, dt + \sigma \cdot \sqrt{v} dW_2 \\
&= \kappa \cdot (\theta - v)\, dt + \sigma \cdot \sqrt{v} \cdot \left(dW_2^* - \lambda_1 \cdot \rho dt - \lambda_2 dt\right) \\
\frac{d\xi}{\xi} &= -\lambda_1 dW_1 - \lambda_2 dW_2
\end{aligned}$$

The state variable (volatility) is not traded! We are make assumption about λ_1 and λ_2; if we assume that $\lambda_2 = 0$, then $\lambda_1 = \frac{\mu - r}{\sqrt{v}}$.

$$\begin{aligned}
dv &= \left\{\kappa \cdot (\theta - v)\, dt - (\mu - r) \cdot \rho \cdot \sigma\right\} dt + \sigma \cdot \sqrt{v} dW_2^* \\
&= \kappa \cdot (\theta^* - v)\, dt + \sigma \cdot \sqrt{v} dW_2^* \\
\theta^* &= \theta - \frac{(\mu - r) \cdot \rho \cdot \sigma}{\kappa} \qquad \theta^* > 0
\end{aligned}$$

If we assume that $\lambda_2 = 0$ we assume that the volatility risk is not priced.

According to Hull&White

$$Q\left(S(T) > K |\, \text{path of } v\right)$$

If the path is known

$$\begin{aligned}
logS(T) &= logS(0) + r \cdot T - \frac{1}{2} \cdot \int_0^T v dt + \int_0^T \sqrt{v} dW_1^* \\
\int_0^T \sqrt{v} dW_1^* &\sim N\left[0, \int_0^T v dt\right]
\end{aligned}$$

Value of a call option:

$$S(0) \cdot \mathbb{E}^{Q^*}\left[N[d_1] |\int_0^T v = \sigma^2 \cdot T\right] - K \cdot e^{-rT} \cdot \mathbb{E}\left[N[d_2] |\int_0^T v = \sigma^2 \cdot T\right]$$

$N[d_1]$ and $N[d_2]$ are close to linear, so we approximate

$$S(0) \cdot N[d_1] - K \cdot e^{-rT} \cdot N[d_2]$$

Where in d_1 put

$$\mathbb{E}^{Q^*}\left[\int_0^T v dt\right] \quad \text{for } \sigma^2 \cdot T$$

and in d_2 put

$$\mathbb{E}^{Q}\left[\int_0^T v dt\right] \quad \text{for } \sigma^2 \cdot T$$

if $S(0)$ is close to K.

8 Risk neutral measure

The money market measure is equal to the risk neutral measure Q.

$$\frac{dQ}{dP} = \rho(T) \cdot \frac{S_1(T)}{S_1(0)}$$

If S_1 is equal to the money market

$$\frac{dQ}{dP} = \rho(T) \cdot e^{\int_0^T r(u)du}$$

Let Q^* be the measure using the stock as numeraire:

$$\frac{dS}{S} = \mu_S(t)\,dt + \sigma_S(t)\,dW_P$$

whereas W_P is a $P-$Brownian motion. Now write S under Q:

$$\frac{dS}{S} = rdt + \sigma_S(t)\,dW$$

whereas W is a $Q-$Brownian motion.

$$\begin{aligned}
\frac{dQ^*}{dP} &= \rho(T) \cdot \frac{S(T)}{S(0)} \\
\frac{dQ^*}{dQ} &= \frac{dQ^*}{dP} \cdot \frac{dP}{dQ} \\
&= \rho(T) \cdot \frac{S(T)}{S(0)} \cdot \frac{1}{\rho(T)} \cdot e^{-\int_0^T r(u)du} \\
&= e^{-\int_0^T r(u)du} \cdot \frac{S(T)}{S(0)}
\end{aligned}$$

Anything divided by the money market account $S(0)$ is a martingale.

$$\begin{aligned}
\xi &= \frac{dQ^*}{dQ} \\
\xi(t) &= \mathbb{E}^Q\left[\xi(T)|\mathcal{F}_t\right] \\
\frac{d\left(e^{-\int_0^t r}\cdot S\right)}{e^{-\int_0^t r}\cdot S} &= -rdt + \frac{dS}{S} \\
&= \sigma_S(t)\,dW \\
\mathbb{E}^Q\left[\left.\frac{e^{\int_0^T rdu}\cdot S(T)}{S(0)}\right|\mathcal{F}_t\right] &= \frac{e^{-\int_0^t rdu}\cdot S(t)}{S(0)} \\
&= \xi(t) \\
\frac{d\xi}{\xi} &= \sigma_S(t)\,dW
\end{aligned}$$

The term $\frac{d\xi}{\xi}$ is always equal to the stochastic part of $\frac{dS}{S}$.

Recipe

1. Write model down under Q
2. Change to different numeraire

$$\frac{d\xi}{\xi} = \text{stochastic part of new numeraire}$$

3. $dW^* = dW - \left(\frac{d\xi}{\xi}\right)dW$

With the stock as numeraire

$$dW^* = dW - \sigma_S dW dW = dW - \sigma_S dt$$

Write S under Q^*

$$\begin{aligned}
\frac{dS}{S} &= rdt + \sigma_S dW \\
&= rdt + \sigma_S\cdot(dW^* + \sigma_S dt) \\
&= \left(r+\sigma_S^2\right)dt + \sigma_S dW^* \\
W^* &\hat{=} Q^* - \text{Brownian Motion}
\end{aligned}$$

9 Heston's model

Under P

$$\begin{aligned}
\frac{dS}{S} &= \mu dt + \sqrt{v} dW_1 \\
dv &= \kappa \cdot (\theta - v)\, dt + \sigma \sqrt{v} dW_2
\end{aligned}$$

Now put $\xi = \frac{dQ}{dP}$ and remember that $\langle W_1, W_2 \rangle = \rho dt$; by martingale representation:

$$\begin{aligned}
\frac{d\xi}{\xi} &= -\lambda_1 dW_1 - \lambda_2 dW_2 \\
dW_1^* &= dW_1 - \frac{d\xi}{\xi} dW_1 \\
&= dW_1 + \lambda_1 dt + \lambda_2 \cdot \rho dt \\
dW_2^* &= dW_2 - \frac{d\xi}{\xi} dW_2 \\
&= dW_2 + \lambda_1 \cdot \rho dt + \lambda_2 dt
\end{aligned}$$

Under the risk neutral measure Q

$$\begin{aligned}
\frac{dS}{S} &= \mu dt + \sqrt{v} \cdot (dW_1^* - \lambda_1 dt - \lambda_2 \cdot \rho dt) \\
&= \left(\mu - \lambda_1 \cdot \sqrt{v} - \lambda_2 \cdot \rho \cdot \sqrt{v}\right) dt + \sqrt{v} dW_1^* \\
\mu - \lambda_1 \cdot \sqrt{v} - \lambda_2 \cdot \rho \cdot \sqrt{v} &= r
\end{aligned}$$

Heston takes

$$(\lambda_1 + \lambda_2 \cdot \rho) \cdot \sigma \cdot \sqrt{v} = \lambda \cdot v$$

whereas λ is constant.

Under Q he writes

$$\begin{aligned}
dv &= \kappa \cdot (\theta - v)\, dt + \sigma \cdot \sqrt{v} \cdot (dW_2^* - \lambda_1 \cdot \rho dt - \lambda_2 dt) \\
&= \kappa \cdot (\theta - v)\, dt - (\lambda_1 \cdot \rho - \lambda_2) \cdot \sigma \cdot \sqrt{v} dt + \sigma \cdot \sqrt{v} dW_2^* \\
&= \kappa \cdot (\theta - v)\, dt - \lambda \cdot v dt + \sigma \cdot \sqrt{v} dW_2^* \\
&= \kappa^* \cdot (\theta^* - v)\, dt + \sigma \cdot \sqrt{v} dW_2^* \\
\kappa^* &= \kappa + \lambda \\
\theta^* &= \frac{\kappa \cdot \theta}{\kappa + \lambda}
\end{aligned}$$

9.1 Heston's model under Q

$$\begin{aligned}
\frac{dS}{S} &= rdt + \sqrt{v}dW_1 \\
dv &= \kappa \cdot (\theta - v)\, dt + \sigma \cdot \sqrt{v}dW_2
\end{aligned}$$

whereas W_1 and W_2 are $Q-$Brownian motions.

The value of an european call option is equal to

$$S(0) \cdot Q^*(S(T) > K) - e^{-rT} \cdot K \cdot Q^*(S(T) > K)$$

where Q^* uses the stock as numeraire. When we change from Q to Q^*

$$\begin{aligned}
\xi &= \frac{dQ^*}{dQ} \\
\frac{d\xi}{\xi} &= \sqrt{v}dW_1
\end{aligned}$$

Under Q^*

$$\begin{aligned}
dW_1^* &= dW_1 - \frac{d\xi}{\xi}dW_1 \\
&= dW_1 - \sqrt{v}dt \\
dW_2^* &= dW_2 - \frac{d\xi}{\xi}dW_2 \\
&= dW_2 - \sqrt{v} \cdot \rho dt \\
\frac{dS}{S} &= (r + v)\, dt + \sqrt{v}dW_1^* \\
dv &= \kappa \cdot (\theta - v)\, dt + \sigma \cdot \rho \cdot vdt + \sigma \cdot \sqrt{v}dW_2^*
\end{aligned}$$

We assume

$$\begin{aligned}
dW_1 dW_2 &= \rho dt \\
d\langle W_1, W_2 \rangle &= \rho dt
\end{aligned}$$

So we can write

$$\begin{aligned}
S(t) &= S(0) \cdot e^{rt - \frac{1}{2} \cdot \int_0^t vdu + \int_0^t \sqrt{v}dW_1} \\
\log S(t) &= \log S(0) + rt - \frac{1}{2} \cdot \int_0^t vdu + \int_0^t \sqrt{v}dW_1 \\
d \log S(t) &= \left(r - \frac{1}{2}v\right) dt + \sqrt{v}dW_1
\end{aligned}$$

Under Q^*

$$\begin{aligned} d\log S(t) &= \left(r + \frac{1}{2}v\right)dt + \sqrt{v}dW_1^* \\ dv &= (\kappa - \sigma \cdot \rho) \cdot \left(\frac{\kappa \cdot \theta}{\kappa - \sigma \cdot \rho} - v\right)dt - \sigma \cdot \sqrt{v}dW_2^* \\ f(x,v,t) &= Q\left(S(T) > K \middle| \log S(t) = x,\ v(t) = v\right) \\ &= \mathbb{E}^Q\left[I_{S(T)>K} \middle| \log S(t) = x,\ v(t) = v\right] \\ f(\log S(t), v(t), t) &\hat{=} Q - \text{martingale} \end{aligned}$$

As a martingale is a process such that

$$M(t) = \mathbb{E}\left[M(T) \middle| \mathcal{F}_t\right]$$

we set

$$\begin{aligned} M(t) &= f(\log S(t), v(t), t) \\ M(T) &= \begin{cases} 1 & \log S(T) > \log K \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

We have defined f so that

$$\begin{aligned} M(t) &= \mathbb{E}^Q\left[M(T) \middle| \mathcal{F}_t\right] \\ df &= \frac{\delta f}{\delta x}d\log S + \frac{\delta f}{\delta v}dv + \frac{\delta f}{\delta t}dt + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta x^2}(d\log S)^2 + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta v^2}(dv)^2 + \frac{\delta^2 f}{\delta x \delta v}(d\log S)(dv) \end{aligned}$$

Drift of df under Q

$$\frac{\delta f}{\delta x} \cdot \left(r - \frac{1}{2} \cdot v\right) + \frac{\delta f}{\delta v} \cdot \kappa \cdot (\theta - v) + \frac{\delta f}{\delta t} + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta x^2} \cdot v + \frac{1}{2} \cdot \frac{\delta^2 f}{\delta v^2} \cdot \sigma \cdot v + \frac{\delta^2 f}{\delta x \delta v} \cdot \sigma \cdot v \cdot \rho = 0$$

So we see that df is a martingale.

We want: $Q(S(T) > K) = f(\log S(0), v(0), 0)$

It is important to check the boundary conditions:

$$\begin{aligned} &\lim_{x\to\infty} f(x,v,t) = 1 \quad \lim_{t\to T} f(x,v,t) = \begin{cases} 1 & x > \log K \\ 0 & x \le \log K \end{cases} \\ &\lim_{x\to 0} f(x,v,t) = 0 \end{aligned}$$

Further we look at $Q^*(S(T) > K)$ and set

$$g(x,v,t) = \mathbb{E}^{Q^*}\left[I_{S(T)>K} \middle| \begin{array}{l} \log S(t) = x \\ \log v(t) = v \end{array}\right]$$

Drift of dg under Q^*

Only the drift changes if we go from Q to Q^*!

$$\frac{\delta g}{\delta x}\cdot\left(r+\frac{1}{2}\cdot v\right)+\frac{\delta g}{\delta v}\cdot\kappa^*\cdot(\theta^*-v)+\frac{\delta g}{\delta t}+\frac{1}{2}\cdot\frac{\delta^2 g}{\delta x^2}\cdot v+\frac{1}{2}\cdot\frac{\delta^2 g}{\delta v^2}\cdot\sigma\cdot v+\frac{\delta^2 g}{\delta x\delta v}\cdot\sigma\cdot\rho\cdot v=0$$

We want the distribution of $S\left(T\right)$ under Q and Q^*. As a characteristic function identifies the distribution it suffices to find this characteristic funtion of $\log S$:

$$ln\Phi=\mathbb{E}^Q\left[e^{i\cdot\Phi\cdot\log S(T)}\right]$$

Consider the following term:

$$\mathbb{E}^Q\left[e^{i\cdot d\log S(T)}\middle|\begin{array}{l}\log S\left(t\right)=x\\ \log v\left(t\right)=v\end{array}\right]=k\left(x,v,t,\Phi\right)$$

For fixed Φ the term $k\left(\log S\left(t\right),v\left(t\right),t,\Phi\right)$ is a $Q-$martingale; it satisfies the PDE (df under Q).

Now we *guess* a solution: $\log k\left(x,v,t,\Phi\right)$ is an affine function of x and v with the parameters depending on t and Φ.

10 Futures contracts

Define $G(t)$ as the futures price for delivery of X at time T, whereas X is in currency units.

If we buy a future at time 0 and sell at time t we make a profit equal to $G(t) - G(0)$. In fact the gain through t is

$$\int_0^t e^{r\cdot(t-s)} dG(s)$$

Under Q:

$$\begin{aligned} e^{-rt} \cdot \int_0^t e^{r\cdot(t-s)} dG(s) &= \int_0^t e^{-r\cdot s} dG(s) \\ d\int_0^t e^{-r\cdot s} dG(s) &= e^{-r\cdot s} dG(s) \end{aligned}$$

Therefore

$$\begin{aligned} G(t) &= \mathbb{E}^Q [X | \mathcal{F}_t] \\ F(t) &= \mathbb{E}^{Q^*} [X | \mathcal{F}_t] \end{aligned}$$

where Q^* is the forward measure using the discount bond as numeraire.

If interest rates are deterministic Q is equal to Q^*, because

$$\begin{aligned} \frac{dQ}{dP} &= \rho(T) \cdot \frac{e^{\int_0^T r}}{1} \\ \frac{dQ^*}{dP} &= \rho(T) \cdot \frac{1}{e^{-\int_0^T r}} \\ e^{-\int_0^T r} &= B(0) \end{aligned}$$

11 Foreign exchange

Define $X(t)$ as the exchange rate, i.e. the price of foreign currency in units of domestic currency. Use Q as the domestic risk neutral measure. For any foreign reinvested asset price S^f the term

$$e^{-\int_0^t r du} \cdot X(t) \cdot S^f(t)$$

is a $Q-$martingale.

Take S^f for the money market account

$$S^f = e^{\int_0^t r^f du}$$

The term

$$e^{\int_0^t \left(r^f - r\right) du} \cdot X(t)$$

is a $Q-$martingale.

$$\frac{d\left(e^{\int_0^t \left(r^f - r\right) du} \cdot X(t)\right)}{e^{\int_0^t \left(r^f - r\right) du} \cdot X(t)} = \left(r^f - r\right) dt + \frac{dX}{X}$$

So the drift of $\frac{dX}{X} = \left(r - r^f\right) dt$ and we see

$$\frac{dX}{X} = \left(r - r^f\right) dt + \sigma_X(t)\, dW_X(t)$$

where W_X is a $Q-$Brownian motion.

$$\begin{aligned} e^{\int_0^t \left(r^f - r\right) du} \cdot X(t) &= \mathbb{E}^Q\left[e^{\int_0^T \left(r^f - r\right) du} \cdot X(T) \middle| \mathcal{F}_t \right] \\ X(t) &= \mathbb{E}^Q\left[e^{\int_t^T \left(r^f - r\right) du} \cdot X(T) \middle| \mathcal{F}_t \right] \end{aligned}$$

Suppose that r^f and r are constant.

$$\begin{aligned} X(t) &= \mathbb{E}^Q\left[e^{\left(r^f - r\right) \cdot (T - t)} \cdot X(T) \middle| \mathcal{F}_t \right] \\ &= e^{\left(r^f - r\right) \cdot (T - t)} \cdot \mathbb{E}^Q\left[X(T) \middle| \mathcal{F}_t \right] \\ &= e^{\left(r^f - r\right) \cdot (T - t)} \cdot G(t) \end{aligned}$$

where $G(t)$ is a covered interest parity.

11.1 Foreign risk neutral measure

We define Q^* as the foreign risk neutral measure, meaning that

$$e^{-\int_0^t r^f du} \cdot S^f(t)$$

is a $Q-$Brownian motion for all reinvested assets S. The domestic price is therefor given as

$$\begin{aligned}
\mathbb{E}\left[\rho(T) \cdot X(T) \cdot S^f(T)\right] &= X(0) \cdot \mathbb{E}\left[\rho(T) \cdot \frac{X(T)}{X(0)} \cdot S^f(T)\right] \\
&= X(0) \cdot \mathbb{E}\left[\rho^f(T) \cdot S^f(T)\right] \\
\rho^f(T) &= \frac{\rho(T) \cdot X(T)}{X(0)}
\end{aligned}$$

So $\frac{1}{X(0)} \cdot$ domestic price $=$ foreign price $= \mathbb{E}\left[\rho^f(T) \cdot S^f(T)\right]$ and ρ^f is a foreign state price density. The foreign risk neutral measure is given by

$$\frac{dQ^*}{dP} = \rho^f(T) \cdot e^{\int_0^T r^f(t)dt}$$

Remember that

$$\frac{dQ}{dP} = \rho(T) \cdot e^{\int_0^T r(t)dt}$$

Therefore

$$\begin{aligned}
\frac{dQ^*}{dQ} &= \frac{dQ^*}{dP} \cdot \frac{dP}{dQ} \\
&= \rho^f \cdot e^{\int_0^T r^f dt} \cdot \frac{1}{\rho(T) \cdot e^{\int_0^T r dt}} \\
&= e^{\int_0^T (r^f - r)dt} \cdot \frac{X(T)}{X(0)}
\end{aligned}$$

Now set

$$\begin{aligned}
\xi &= \frac{dQ^*}{dQ} \\
\xi(t) &= \mathbb{E}^Q\left[\xi(T) | \mathcal{F}_t\right] \\
&= \mathbb{E}^Q\left[e^{\int_0^T (r^f - r)dt} \cdot \frac{X(T)}{X(0)} \middle| \mathcal{F}_t\right] \\
&= e^{\int_0^t (r^f - r)du} \cdot \frac{X(t)}{X(0)}
\end{aligned}$$

As the term

$$\frac{d\xi}{\xi} = \left(r^f - r\right) dt + \frac{dX}{X}$$

can't have a drift, $\frac{d\xi}{\xi}$ is the stochastic part of $\frac{dX}{X}$

$$\frac{d\xi}{\xi} = \sigma_X(t) \, dW_X(t)$$

12 Quanto forward

We are using the same pricing as in chapter 11.1 with the foreign asset S^f and the forward price K in foreign currency. We fix the exchange rate at maturity as $\bar{X}$, so the contract pays $\bar{X} \cdot \left(S^f(T) - K\right)$ in domestic currency at time T.

12.1 Straight forward on a foreign asset

Pays $\bar{X} \cdot \left(S^f(T) - K\right)$ in domestic currency, where $\bar{X}$ is a contract-fixed exchange rate.

Start under Q:

$$\frac{dX}{X} = \left(r - r^f\right) dt + \sigma_X dW_X$$

where W_X is a $Q-$Brownian motion. Know that

$$e^{-rt} \cdot X(t) \cdot S^f(t)$$

is $Q-$martingal.

$$\begin{aligned}
\frac{dS^f}{S^f} &= \mu_S dt + \sigma_S dW_S \\
\frac{d\left(e^{-rt} \cdot X(t) \cdot S^f(t)\right)}{e^{-rt} \cdot X(t) \cdot S^f(t)} &= -rdt + \underbrace{\frac{dX}{X} + \frac{dS^f}{S^f} + \frac{d\left\langle X, S^f \right\rangle}{X \cdot S^f}}_{\text{must be } rdt} \\
&= -rdt + \left(r - r^f\right) dt + \sigma_X dW_X + \mu_S dt + \sigma_S dW_S + \sigma_X \cdot \sigma_S \cdot \rho dt
\end{aligned}$$

where ρ is the correlation process of W_X and W_S.

The drift of $\frac{d\left(X \cdot S^f\right)}{X \cdot S^f}$ must be rdt.

$$\begin{aligned}
\frac{d\left(X \cdot S^f\right)}{X \cdot S^f} &= \frac{dX}{X} + \frac{dS^f}{S^f} + \frac{d\left\langle X, S^f \right\rangle}{X \cdot S^f} \\
&= \left(r - r^f\right) dt + \sigma_X dW_X + \mu_S dt + \sigma_S dW_S + \sigma_X \cdot \sigma_S \cdot \rho dt
\end{aligned}$$

Now compute the drift:

$$\left(r - r^f + \mu_S + \sigma_X \cdot \sigma_S \cdot \rho\right) = r$$

It follows that

$$\mu_S = r_f - \sigma_X \cdot \sigma_S \cdot \rho$$

and so

$$\frac{dS^f}{S^f} = (r_f - \sigma_X \cdot \sigma_S \cdot \rho)\, dt + \sigma_S dW_S$$

So we see that the drift of $\frac{dS^f}{S^f}$ is equal to the risk free rate minus the correlation of $\frac{dX}{X}$ with $\frac{dS^f}{S^f}$.

Now compute

$$\begin{aligned} e^{-rT} \cdot \mathbb{E}^Q\left[\bar{X} \cdot \left(S^f(T) - K\right)\right] &= e^{-rT} \cdot \bar{X} \cdot \mathbb{E}^Q\left[S^f(T)\right] - e^{-rT} \cdot \bar{X} \cdot K \\ S^f(T) &= S^f(0) \cdot e^{\left(r_f - \sigma_S \cdot \sigma_X \cdot \rho - \frac{1}{2} \cdot \sigma_S^2\right) \cdot T + \sigma_S \cdot W_S(T)} \\ \mathbb{E}^Q\left[S^f(T)\right] &= S^f(0) \cdot e^{\left(r_f - \sigma_S \cdot \sigma_X \cdot \rho\right) \cdot T} \end{aligned}$$

12.1.1 Wrong calculation (ignore this)

Pays $S^f(T) - K$ in foreign currency at time T. We start with a domestic point of view; the value of this contract is equal to

$$\begin{aligned} e^{-rT} \cdot \mathbb{E}^Q\left[X(T) \cdot \left[S^f(T) \cdot K\right]\right] &= e^{-rT} \cdot \mathbb{E}^Q\left[X(T) \cdot S^f(T)\right] - e^{-rT} \cdot \mathbb{E}^Q\left[X(T)\right] \cdot K \\ &= e^{-rT} \cdot X(0) \cdot S^f(0) - K \cdot e^{-rT} \cdot \text{futures rate} \end{aligned}$$

For this to be fairly priced K must be equal to

$$K = \frac{\text{spot rate}}{\text{futures rate}} \cdot S^f(0)$$

Under the $Q-$domestic risk neutral measure the value must be

$$e^{-rT} \cdot \mathbb{E}^Q\left[\bar{X} \cdot \left(S^f(T) - K\right)\right] = e^{-rT} \cdot \bar{X} \cdot \mathbb{E}^Q\left[S^f(T)\right] - e^{-rT} \cdot \bar{X} \cdot K$$

Under the Q^*-foreign risk neutral measure

$$\begin{aligned}
\frac{dS^f}{S^f} &= r^f dt + \sigma_S dW_S \\
\frac{dQ^*}{dQ} &= e^{\left(r^f - r\right)\cdot T} \cdot \frac{X(T)}{X(0)} \\
\frac{dQ}{dQ^*} &= e^{\left(r - r^f\right)\cdot T} \cdot \frac{X(0)}{X(T)} \\
\xi &= \frac{dQ}{dQ^*} \\
\xi &= \mathbb{E}^{Q^*}\left[\xi | \mathcal{F}_t\right] \\
&= e^{\left(r - r^f\right)} \frac{X(0)}{X(T)} \\
\frac{d\xi}{\xi} &= \left(r - r^f\right) dt - \frac{dX}{X} + \frac{d\langle X, X\rangle}{X^2} \\
dW_S^* &= dW_S - \frac{d\xi}{\xi} dW_S \\
&= dW_S - \sigma_X dW_X dW_S \\
&= dW_S - \sigma_X \cdot \gamma dt \\
\gamma &= d\langle W_X, W_S\rangle
\end{aligned}$$

Further

$$\begin{aligned}
\frac{dS^f}{S^f} &= r^f dt + \sigma_S \left(dW_S^* + \sigma_X \cdot \gamma dt\right) \\
&= \left(r^f + \sigma_X \cdot \sigma_S \cdot \gamma\right) dt + \sigma_S dW_S^* \\
S^f(T) &= S^f(0) \cdot e^{\left(r^f + \sigma_X \cdot \sigma_S \cdot \gamma - \frac{1}{2}\cdot \sigma_S^2\right)\cdot T + \sigma_S \cdot W_S^*(T)} \\
\mathbb{E}^Q\left[S^f(T)\right] &= S^f(0) \cdot e^{\left(r^f + \sigma_X \cdot \sigma_S \cdot \gamma\right)\cdot T}
\end{aligned}$$

THIS IS WRONG! WRONG SIGN!

AD PROBLEMS:

$$\begin{aligned}
\left[K - Z(T)\right]^+ &= K \cdot Z(T) \cdot \left[\frac{1}{Z(T)} - \frac{1}{K}\right]^+ \\
\mathbb{E}_t\left[X_T\right] &= \mathbb{E}\left[X_t | \mathcal{F}_t\right]
\end{aligned}$$

12.2 Straight foreign forward

Forward contract on a foreign asset (no dividends) with price in foreign currency ($K =$forward price).

Under the domestic risk neutral measure, where $X \cdot S^f$ is a domestic asset, we want

$$\mathbb{E}^Q\left[X(T) \cdot \left[S^f(T) - K\right]\right] = 0$$

Compute $\mathbb{E}^Q\left[X(T) \cdot S^f(T)\right]$

Because

$$e^{-rt} \cdot X(t) \cdot S^f(t)$$

is a $Q-$martingale it follows

$$\begin{aligned} X(0) \cdot S^f(0) &= \mathbb{E}^Q\left[e^{-rT} \cdot X(T) \cdot S^f(T)\right] \\ e^{rT} \cdot X(0) \cdot S^f(0) &= \mathbb{E}^Q\left[X(T) \cdot S^f(T)\right] \end{aligned}$$

Compute $\mathbb{E}^Q[X(T) \cdot K]$

$$\begin{aligned} \mathbb{E}^Q[X(T) \cdot K] &= K \cdot \mathbb{E}^Q[X(T)] \\ &= K \cdot G(0) \end{aligned}$$

where $G(0)$ is the currency futures price at time 0.

Covered interest parity:

$$\begin{aligned} X(0) &= e^{\left(r^f - r\right) \cdot T} \cdot G(0) \\ \mathbb{E}^Q[X(T) \cdot K] &= K \cdot e^{\left(r - r^f\right) \cdot T} \cdot X(0) \end{aligned}$$

Now we know

$$\begin{aligned} e^{rT} \cdot X(0) \cdot S^f(0) - K \cdot e^{\left(r - r^f\right) \cdot T} \cdot X(0) &= 0 \\ K &= e^{rT} \cdot S^f(0) \end{aligned}$$

12.3 Synthetic Forward

Construct a portfolio, where we buy the asset and borrow $S^f(0)$ in foreign currency: we owe $e^{rT} \cdot S^f(0)$ at time T.

Start under the domestic risk neutral measure Q.

1. Stock as numeraire Q^*

$$\frac{dS}{S} = rdt + \sigma_S \cdot dW_s \quad \text{under } Q$$

$$\xi = \frac{dQ^*}{dQ} \qquad \frac{d\xi}{\xi} = \sigma_S dW_S$$

For any other $Q-$Brownian motion W in the model we define a Q^*-Brownian motion as

$$\begin{aligned} dW^* &= dW - \left(\frac{dS}{\xi}\right)(dW) \\ &= dW - \sigma_S (dW_s)(dW) \\ &= dW - \sigma_S \cdot \gamma dt \end{aligned}$$

for some γ, which describes the relation to the stock.

2. Forward measure Q^*(uses discount bond as numeraire)

$$\frac{dB}{B} = rdt + \sigma_B dW_B \quad \text{under } Q$$

$$\xi = \frac{dQ^*}{dQ} \qquad \frac{d\xi}{\xi} = \sigma_B dW_B$$

For any other $Q-$Brownian motion W we define a Q^*-Brownian motion as

$$\begin{aligned} dW^* &= dW - \left(\frac{dS}{S}\right) dW \\ &= dW - \sigma_B \cdot \gamma dt \end{aligned}$$

for some γ.

3. Foreign risk neutral measure Q^*

$$e^{-r^f t} \cdot S^f(t)$$

is a Q^*-martingal for any foreign reinvested asset price S^f. Under Q

$$\frac{dX}{X} = \left(r - r^f\right) dt - \sigma_X dW_x$$

$$\xi = \frac{dQ^*}{dQ} \qquad \frac{d\xi}{\xi} = \sigma_X dW_X$$

For any other $Q-$Brownian motion W we define a Q^*-Brownian motion as

$$dW^* = dW - \sigma_X \cdot \gamma dt$$

for some γ.

Given two Brownian motions, W_1 and W_2,

$$\begin{aligned} d(W_1)(W_2) &= d\langle W_1, W_2\rangle \\ \langle W_1, W_2\rangle(t) &= \lim_{\text{partitions}} \sum_1^n (\Delta W_1)(\Delta W_2) \end{aligned}$$

Then the term

$$W_1(t) W_2(t) - \langle W_1, W_2\rangle(t)$$

is a martingale.

Suppose:

$$\begin{aligned} cov\left(W_1(t), W_2(t)\right) &= \mathbb{E}\left[W_1(t) \cdot W_2(t)\right] \\ &= \underbrace{\mathbb{E}\left[W_1(t)\right]}_{0} \cdot \underbrace{\mathbb{E}\left[W_2(t)\right]}_{0} \end{aligned}$$

For some ρ

$$d(W_1) \cdot d(W_2) = \rho dt$$

This means

$$\begin{aligned} d\langle W_1, W_2\rangle(t) &= \rho dt \\ \langle W_1, W_2\rangle(t) &= \underbrace{\langle W_1, W_2\rangle(0)}_{0} + \int_0^t \rho(s)\, ds \end{aligned}$$

So the term

$$W_1(t) \cdot W_2(t) - \int_0^t \rho(s)\, ds$$

is a martingale.

$$\begin{aligned} \mathbb{E}\left[W_1(t) \cdot W_2(t) - \int_0^t \rho(s)\, ds\right] &= 0 \\ cov\left(W_1(t), W_2(t)\right) &= \mathbb{E}\left[\int_0^t \rho(s)\, ds\right] \end{aligned}$$

As the standard deviation of $W(t)$ equals $\sqrt{t}$

$$\begin{aligned} corr\left(W_1(t), W_2(t)\right) &= \frac{cov\left(W_1(t), W_2(t)\right)}{\sigma_1 \cdot \sigma_2} \\ &= \mathbb{E}\left[\frac{1}{t}\int_0^t \rho(s)\, ds\right] \end{aligned}$$

Now consider

$$\begin{aligned} dM_1 &= \sigma_1 dW_1 \\ dM_2 &= \sigma_2 dW_2 \end{aligned}$$

This means

$$\begin{aligned} M_1(t) &= M_1(0) + \int_0^t \sigma_1(s)\, dW_1(s) \\ M_2(t) &= M_2(0) + \int_0^t \sigma_2(s)\, dW_2(s) \end{aligned}$$

Write

$$(dM_1) \cdot (dM_2) - \sigma_1 \cdot \sigma_2 \rho dt$$

where ρ is the correlation process for W_1 and W_2.

This means

$$\begin{aligned} d \langle M_1, M_2 \rangle &= \sigma_1 \cdot \sigma_2 \cdot \rho dt \\ \langle M_1, M_2 \rangle (t) &= \int_0^t \sigma_1 (s) \cdot \sigma_2 (s) \rho (s) \, ds \end{aligned}$$

and so

$$M_1 \cdot M_2 - \langle M_1, M_2 \rangle$$

is a martingale.

$$\begin{aligned} cov (M_1 (t), M_2 (t)) &= \mathbb{E} \left[\int_0^t \sigma_1 (s) \cdot \sigma_2 (s) \cdot \rho (s) \, ds \right] \\ var (M_1 (t)) &= \mathbb{E} \left[\int_0^t \sigma_1^2 (s) \, ds \right] \end{aligned}$$

13 Review stock options with proportional dividend yields

Let Z be the asset price and δ a constant proportional dividend yield:

$$\begin{aligned} dD &= \delta \cdot Z \\ S(t) &= e^{\delta t} \cdot Z(t) \end{aligned}$$

Now compute

$$\begin{aligned} e^{-rT} \cdot \mathbb{E}^Q \left[(Z(T) - K)^t \right] &= e^{-rT} \cdot \mathbb{E}^Q \left[Z(T) \cdot I_{Z(T)>K} \right] - e^{-rT} \cdot K \cdot \mathbb{E}^Q \left[I_{Z(T)>K} \right] \\ \frac{dS}{S} &= r dt + \sigma dW \\ \frac{dZ}{Z} &= (r - \delta) dt + \sigma dW \\ \mathbb{E}^Q \left[I_{Z(T)>K} \right] &= Q(Z(T) > K) \\ &= N[d_2] \\ d_2 &= \frac{\log \frac{S(0)}{K} + \left(r - \delta - \frac{1}{2} \cdot \sigma^2 \right)}{\sigma \cdot \sqrt{T}} \\ \mathbb{E}^Q \left[Z(T) \cdot I_{Z(T)>K} \right] &= e^{-\delta T} \cdot \mathbb{E}^Q \left[S(T) \cdot I_{Z(T)>K} \right] \\ &= e^{(r-\delta)T} \cdot S(0) \cdot \mathbb{E}^Q \left[\frac{e^{-rT} \cdot S(T)}{S(0)} \cdot I_{Z(T)>K} \right] \\ &= e^{(r-\delta)T} \cdot S(0) \cdot \mathbb{E}^Q \left[\frac{dQ^*}{dQ} \cdot I_{Z(T)>K} \right] \\ &= e^{(r-\delta)T} \cdot S(0) \cdot \mathbb{E}^{Q^*} \left[I_{Z(T)>K} \right] \\ &= e^{(r-\delta)T} \cdot S(0) \cdot Q^* (Z(T) > K) \end{aligned}$$

Switching from Q to Q^* adds $\sigma^2 dt$ to the drift of $\frac{dZ}{Z}$:

$$\begin{aligned} e^{-rT} \cdot \mathbb{E}^Q \left[Z(T) \cdot I_{Z(T)>K} \right] &= e^{-\delta T} \cdot S(0) \cdot N[d_1] \\ d_1 &= \frac{\log \frac{S(0)}{K} + \left(r - \delta + \frac{1}{2} \cdot \sigma^2 \right) \cdot T}{\sigma \cdot \sqrt{T}} \end{aligned}$$

So the formula for a call equals to:

$$e^{-\delta T} \cdot S(0) \cdot N[d_1] - e^{-rT} \cdot K \cdot N[d_2]$$

13.1 Quanto option (no dividends)

Pays $\bar{X}\cdot\left(S^{f}\left(T\right)-K\right)^{+}$. The option value is equal to

$$\begin{aligned}
-e^{rT}\cdot\mathbb{E}^{Q}\left[\bar{X}\cdot\left(S^{f}\left(T\right)-K\right)^{+}\right] &= e^{-rT}\cdot\bar{X}\cdot\mathbb{E}^{Q}\left[\left(S^{f}\left(T\right)-K\right)^{+}\right]\\
\frac{dS^{f}}{S^{f}} &= \left(r^{f}-\sigma_{X}\cdot\sigma_{S}\cdot\rho\right)dt+\sigma_{S}dW_{S}\\
&= \left(r-\delta\right)dt+\sigma_{S}dW_{S}\\
\delta &= r+\sigma_{X}\cdot\sigma_{S}\cdot\rho-r^{f}
\end{aligned}$$

So we can compute the value of the quanto option as:

$$\bar{X}\cdot e^{-\delta T}\cdot S\left(0\right)\cdot N\left[d_{1}\right]-\bar{X}\cdot e^{-rT}\cdot K\cdot N\left[d_{2}\right]$$

13.2 American put option

The optimal exercise time is the first time that $S\left(t\right)$ drops to the optimal exercise boundary $b\left(t\right)$, which is a monotonic increasing function with $\lim_{t\to T}K$, i.e the exercise accelerate the receipt of K. The value of an american put option is equal to

$$\sup_{\tau}\mathbb{E}^{Q}\left[e^{-r\tau}\cdot\left(K-S\left(\tau\right)\right)^{+}\right]=\mathbb{E}\left[e^{-r\tau^{*}}\cdot\left(K-S\left(\tau^{*}\right)\right)^{+}\right]$$

where τ is the exercise time and $\tau\left(\omega\right)=t$ means exercise at t in state ω of the world and $\tau^{*}=\inf\left\{t\middle|\,S\left(t\right)\leq b\left(t\right)\right\}$. We can think of the value of an american put options as the value of an european put option plus some kind of early exercise premium:

$$\mathbb{E}^{Q}\left[\int_{0}^{T}e^{-ru}\cdot I_{S(u)\leq b(u)}\cdot r\cdot K du\right]$$

Now set $P\left(x,t\right)$ equal the put value at time t where $S\left(t\right)=x$. We know that $P\left(x,t\right)=K-x$ when $x\leq b\left(t\right)$. If $x>b\left(t\right)$ then $P\left(x,t\right)$ is equal to the european value at t when $S\left(t\right)=x+$ exercise premium:

$$\mathbb{E}^{Q}\left[\int_{t}^{T}e^{-r(u-t)}\cdot I_{S(u)\leq b(u)}\cdot r\cdot K du\middle|\,S\left(t\right)=x\right]$$

Our value matching condition is

$$\begin{aligned}
K-b\left(t\right) &= \{\text{european put value at } t \text{ when } S\left(t\right)=b\left(t\right)\}+\\
&\quad+\mathbb{E}^{Q}\left[\int_{t}^{T}e^{-r(u-t)}\cdot I_{S(u)\leq b(u)}\cdot r\cdot K du\middle|\,S\left(t\right)=b\left(t\right)\right]
\end{aligned}$$

which gives us the integral condition to be solved for the function b (see Broadie and Detemple, Review of financial studies, 1996 for a fast numerical procedure).

The early exercise premium at t is equal to

$$\int_t^T r{\cdot}K{\cdot}e^{-r(u-t)}{\cdot}Q\left(S(u)\le b(u)|\,S(t)\right)=\int_t^T r{\cdot}K{\cdot}e^{-r(u-t)}{\cdot}N\left[d\left(u-t,S(t),\underbrace{b(u)}_{\text{role of K}}\right)\right]$$

Now consider different constant exercise boundaries:

$$\begin{aligned}
\frac{dS}{S} &= rdt+\sigma dW \\
S(u) &= S(t)\cdot e^{\left(r-\frac{1}{2}\sigma^2\right)\cdot(u-t)+\sigma[\omega(u)-\omega(t)]} \\
\log S(u) &= \log S(t)+\left(r-\frac{1}{2}\sigma^2\right)\cdot(u-t)+\sigma\left[\omega(u)-\omega(t)\right] \\
Q\left(\log S(u)<\log b(u)\right) &= Q\left(\log S(t)+\left(r-\frac{1}{2}\sigma^2\right)\cdot(u-t)+\sigma\left(\omega(u)-\omega(t)\right)<\log b(u)\right) \\
&= Q\left(\frac{\omega(u)-\omega(t)}{\sqrt{u-t}}\le\underbrace{\frac{\log\frac{b(u)}{S(t)}-\left(r-\frac{1}{2}\sigma^2\right)\cdot(u-t)}{\sigma\cdot\sqrt{u-t}}}_{d(u-t,s(t),b(u))}\right)
\end{aligned}$$

How to find the lower boundary

Dont work with optimal boundaries but assume constant (i.e. flat lines) boundaries.

1. Find the best flat line (simpler than curved lines).
2. Hypothetically move this line down and start again.
3. End up closely above intersecting b.
4. Find this point for every t.

That is the lower boundary.

Now find boundary $c(t)$ that is slightly above $b(t)$:

$$\int_t^T r\cdot K\cdot e^{-r(u-t)}\cdot N\left[d\left(u-t\right),S(t),c(u)\right]>\text{true early exercise premium}$$

13.2.1 Lower bound on put value

Consider a flat boundary X starting at t. Now calculate

$$\mathbb{E}\left[e^{-r(T-t)}\left(K-S\left(\tau\right)\right)^{+}\middle|\,S\left(t\right)\right]$$

where

$$\tau=\inf\left\{t|\,S\left(t\right)\leq X\right\}$$

The hitting time of a flat line by a brownian motion has a known distribution. If we maximize over X we get the lower bound.

14 Replicating value process

Let the assets Y and Z be reinvested assets. Under Q we have

$$\begin{aligned} \frac{dY}{Y} &= rdt + \sigma_Y dW_Y \\ \frac{dZ}{Z} &= rdt + \sigma_Z dW_Z \end{aligned}$$

Let V be the value of a self financing contingent claim, e.g.

$$V(t) = e^{-r\cdot(T-t)} \cdot \mathbb{E}^Q \left[X \middle| \mathcal{F}_t \right]$$

Suppose that V can be replicated by using Y and Z, i.e.

$$V(t) = f(t, Y(t), Z(t)) \qquad \text{for some } f$$

Generally we have

$$\frac{dV}{V} = rdt + \text{stochastic part}$$

According to Ito's Lemma:

1. Derive fundamental PDE (Drift of $df = r \cdot f dt$)
2. The stochastic part of dV is equal to $\frac{\delta f}{\delta Z}$ $\cdot$stochastic part of $dY + \frac{\delta f}{\delta Z}$ $\cdot$stochastic part of dZ.

It follows:

$$\begin{aligned} df &= r^f dt + \frac{\delta f}{\delta Y} \cdot (dY - r \cdot Y dt) + \frac{\delta f}{\delta Z} \cdot (dZ - r \cdot Z dt) \\ &= \frac{\delta f}{\delta Y} dY + \frac{\delta f}{\delta Z} dZ + \left(f - \frac{\delta f}{\delta Y} \cdot Y - \frac{\delta f}{\delta Z} \cdot Z \right) \cdot rdt \end{aligned}$$

Now build a portfolio with $\frac{\delta f}{\delta Y}$ shares of Y and $\frac{\delta f}{\delta Z}$ shares of Z and $f - \frac{\delta f}{\delta Y} - \frac{\delta f}{\delta Z}$ in the money market, which will replicate f, i.e. change in value of portfolio equals df.

14.1 Hedge of Quanto forward

The contract pays $\bar{X} \cdot \left(S^f(T) - K\right)$ at time T. The fair price K is such that the contract has value 0 at time 0:

$$\begin{aligned} e^{-rT} \cdot \mathbb{E}^Q\left[\bar{X} \cdot \left(S^f(T) - K\right)\right] &= 0 \\ \mathbb{E}^Q\left[S^f(T)\right] &= K \\ \frac{dS^f}{S^f} &= \left(r^f - \sigma_X \cdot \sigma_S \cdot \rho\right) dt + \sigma_S dW_S \end{aligned}$$

Now consider $X \cdot S^f$:

$$\begin{aligned} \frac{d\left(X \cdot S^f\right)}{X \cdot S^f} &= rdt + \text{stochastic part} \\ \frac{dX}{X} + \frac{dS^f}{S^f} + \frac{d\left\langle X, S^f\right\rangle}{X \cdot S^f} &= rdt + \text{stochastic part} \\ &= \left(r - r^f\right) dt + \frac{dS^f}{S^f} + \sigma_X \cdot \sigma_S \cdot \rho dt \\ r &= r - r^f + \sigma_X \cdot \sigma_S \cdot \rho + \left\{dt - \text{part of } \frac{dS^f}{S^f}\right\} \\ S^f(T) &= S^f(0) \cdot e^{\left(r^f - \sigma_X \cdot \sigma_S \cdot \rho - \frac{1}{2} \cdot \sigma_S^2\right) \cdot T + \sigma_S \cdot W_S(T)} \\ \mathbb{E}^Q\left[S^f(T)\right] &= S^f(0) \cdot e^{\left(r^f - \sigma_X \cdot \sigma_S \cdot \rho\right) \cdot T} \\ &= \text{fair value at time } 0 \end{aligned}$$

The fair value at time t must be equal to

$$F(t) = S^f(t) \cdot e^{\left(r^f - \sigma_X \cdot \sigma_S \cdot \rho\right) \cdot (T-t)}$$

Now consider a long forward at time 0 at price $F(0)$. The value at time t is equal to:

$$e^{-r(T-t)} \cdot \mathbb{E}^Q\left[\bar{X} \cdot (F(t) - F(0))\right] = e^{-r(T-t)}$$

If you sell a forward at time 0 and you want to hedge, then you need a portfolio with value

$$\begin{aligned} e^{-r(T-t)} \cdot \bar{X} \cdot (F(t) - F(0)) &= e^{-r(T-t)} \cdot \bar{X} \cdot \left(e^{\left(r^f - \sigma_X \cdot \sigma_S \cdot \rho\right) \cdot (T-t)} \cdot S^f(t) - F(0)\right) \\ &= e^{\left(-r + r^f - \sigma_X \cdot \sigma_S \cdot \rho\right) \cdot (T-t)} \cdot \bar{X} \cdot S^f(t) - e^{-r(T-t)} \cdot \bar{X} \cdot F(0) \end{aligned}$$

As S^f is non-domestic it is tricky to calculate. We set

$$\begin{aligned} Y(t) &= X(t) \cdot S^f(t) \\ Z(t) &= e^{r^f \cdot t} \cdot X(t) \end{aligned}$$

It follows

$$\begin{aligned} \frac{Y}{Z} &= e^{-r^f \cdot t} \cdot S^f(t) \\ S^f(t) &= e^{r^f \cdot t} \cdot \frac{Y(t)}{Z(t)} \end{aligned}$$

Now set

$$\begin{aligned} g(t, Y, Z) &= e^{\left(-r+r^f-\sigma_X \cdot \sigma_S \cdot \rho\right)\cdot(T-t)} \cdot \bar{X} \cdot e^{r^f \cdot T} \cdot \frac{Y}{Z} - e^{-r(T-t)} \cdot \bar{X} \cdot F(0) \\ &= e^{\left(-r+r^f-\sigma_X \cdot \sigma_S \cdot \rho\right)\cdot(T-t)} \cdot \bar{X} \cdot S^f(t) - e^{-r(T-t)} \cdot \bar{X} \cdot F(0) \end{aligned}$$

Then g is the value process we want to replicate.

$$\begin{aligned} \frac{\delta g}{\delta Y} &= \text{number of shares of } Y \text{ to hold} \\ \frac{\delta g}{\delta Z} &= \text{number of shares of } Z \text{ to hold} \end{aligned}$$

Example: $\frac{\delta g}{\delta Y} = Z$

$$\begin{aligned} \frac{\delta g}{\delta Y} \cdot Y &= \text{value of } Y \text{ shares in domestic currency} \\ g - \frac{\delta g}{\delta Y} \cdot Y - \frac{\delta g}{\delta Z} \cdot Z &= \text{invested in domestic money market} \\ \frac{\delta g}{\delta Y} &= e^{\left(-r+r^f-\sigma_X \cdot \sigma_S \cdot \rho\right)\cdot(T-t)} \cdot \bar{X} \cdot e^{r^f \cdot t} \cdot \frac{1}{e^{r^f \cdot t}\left(X(t)\right)} \\ &= \frac{\bar{X}}{X(t)} \cdot e^{\left(-r+r^f+\sigma_X \cdot \sigma_S \cdot \rho\right)\cdot(T-t)} \\ \frac{\delta g}{\delta Y} \cdot Y(t) &= e^{\left(-r+r^f-\sigma_X \cdot \sigma_S \cdot \rho\right)\cdot(T-t)} \cdot \bar{X} \cdot S^f(t) \\ \frac{\delta g}{\delta Y} &= \text{number of shares of foreign asset to hold} \\ \frac{\delta g}{\delta Y} \cdot X(t) \cdot S^f(t) &= \frac{\delta g}{\delta Y} \cdot Y(t) \qquad \text{value of foreign asset in domestic currency} \\ \frac{\delta g}{\delta Z} \cdot Z(t) &= \frac{\delta g}{\delta Z} \cdot e^{r^f \cdot t} \cdot X(t) \\ &= \text{Value in domestic currency of position in foreign money market} \end{aligned}$$

15 Examples

15.1 Example 1

Consider e european put option on a stock with constant proportional dividend yield δ and constant volatility coefficient σ. Assume the risk-free rate is constant. The payoff of the put, with exercise price K, is

$$[K - Z(T)]^+ = K \cdot Z(T) \cdot \left[\frac{1}{Z(T)} - \frac{1}{K}\right]^+$$

The interpretation is that exchanging one share of K units of currency is equivalent to buying the currency at a price equal to one share, or equivalent to K purchases of a currency unit, each at cost $\frac{1}{K}$ when measured in shares. The "price" of a currency unit in shares is of course $\frac{1}{Z(T)}$. Use this formula and a change of measure to show that the value of the put is given by the Black-Scholes *call* option formulae when we interchange $Z(0)$ with K and interchange r with δ in the formula. Note that this is not the usual put-call parity!

Solution:

At time 0 we have

$$\begin{aligned}\mathbb{E}\left[\gamma(T) \cdot K \cdot Z(T) \cdot \left[\frac{1}{Z(T)} - \frac{1}{K}\right]^+\right] &= \underbrace{\mathbb{E}\left[\gamma(T) \cdot K \cdot Z(T) \cdot \frac{1}{Z(T)} \cdot I_{\frac{1}{Z(T)} > \frac{1}{K}}\right]}_{\clubsuit} - \\ &\quad - \underbrace{\mathbb{E}\left[S(T) \cdot K \cdot Z(T) \cdot \frac{1}{K} \cdot I_{\frac{1}{Z(T)} > \frac{1}{K}}\right]}_{\diamondsuit}\end{aligned}$$

Using the discount bound as numeraire ($S_1 \widehat{=}$money market) we get for $\clubsuit$:

$$S_1(0) \cdot \mathbb{E}\left[\gamma(T) \cdot \frac{S_1(T)}{S_1(0)} \cdot K \cdot I_{Z(T)<K}\right] = K \cdot S_1(0) \cdot Q(Z(T) < K)$$

And with F for the forward price of the stock and with a long forward on the stock together with $F(0)$ discount bonds as numeraire we get for $\diamondsuit$:

$$F(0) \cdot S_1(0) \cdot Q^*(Z(T) < K)$$

$$
\begin{aligned}
Q\left(Z\left(T\right)<K\right) &= 1-Q\left(Z\left(T\right)\geq K\right)\\
&= N\left[-d_2\right]\\
&= N\left[\frac{\log\frac{K}{S(0)}+\left(\delta-r+\frac{1}{2}\cdot\sigma^2\right)\cdot T}{\sigma\cdot\sqrt{T}}\right]\\
Q^*\left(Z\left(T\right)<K\right) &= N\left[-d_1\right]\\
&= N\left[\frac{\log\frac{K}{S(0)}+\left(\delta-r-\frac{1}{2}\cdot\sigma^2\right)\cdot T}{\sigma\cdot\sqrt{T}}\right]
\end{aligned}
$$

15.2 Example 2

Suppose the domestic and foreign interest rates are constant and equal.

1. Show that under the domestic risk-neutral measure Q, $\mathbb{E}_t^Q\left[X\left(T\right)\middle|\mathcal{F}_t\right]=X\left(t\right)$, where X is the exchange rate (price of foreign currency in units of domestic currency). Start from the most primitive assumptions you can.

   Solution:

$$
\begin{aligned}
S^f\left(t\right)\cdot X\left(t\right)\cdot e^{-\int_0^t r du} &\hat{=} Q-\text{martingal}\\
S^f\left(t\right)=e^{-\int_0^t r^f du}X\left(t\right)\cdot e^{\int_0^t\left(r_f-r\right)du} &= X\left(t\right)
\end{aligned}
$$

2. Show that

$$\mathbb{E}_t^Q\left[\frac{1}{X\left(T\right)}\right]\neq\frac{1}{X\left(t\right)}$$

   unless X is constant.

   Solution:

   Under Q is must be

$$\frac{dX}{X}=0\cdot dt+\sigma_X\left(t\right)dW$$

   Now we use Ito with $f\left(t,X\right)=\frac{1}{X}$

$$\frac{\delta f}{\delta t}=0\qquad\frac{\delta f}{\delta X}=-\frac{1}{X^2}\qquad\frac{\delta^2 f}{\delta X^2}=2\cdot\frac{1}{X^3}$$

$$
\begin{aligned}
df &= \left(\frac{1}{2}\cdot\sigma_X^2\left(t\right)\cdot X^2\cdot 2\cdot\frac{1}{X^3}\right)dt+\sigma_x\left(t\right)\cdot X\cdot\frac{-1}{X^2}dW\\
&= \sigma_X^2\cdot\frac{1}{X}dt-\sigma_X\left(t\right)\cdot\frac{1}{X}dW\\
\frac{d\frac{1}{X}}{\frac{1}{X}} &= \sigma_X^2\left(t\right)dt-\sigma_X\left(t\right)dW
\end{aligned}
$$

So we see that the drift of $\frac{d\frac{1}{X}}{\frac{1}{X}}$ is zero iff X is constant, i.e. $\sigma_X^2(t)$ is zero.

15.3 Example 3

15.4 Example 4

A supershare pays $\frac{S(T)}{K}$, when $K_1 \leq S(T) \leq K_2$ and nothing otherwise, where K_1 and K_2 are given. What is the fair value of a supershare if the interest rate is constant and S is a non-dividend paying asset with constant volatility? What if S pays a constant proportional dividend yield?

Solution:

1. Without dividend

$$\begin{aligned} V &= e^{-rT} \cdot \mathbb{E}^Q \left[\frac{S(T)}{K_1} \cdot I_{K_1 \leq S(T) \leq K_2} \right] \\ &= \frac{e^{-rT}}{K_1} \cdot \mathbb{E}^Q \left[S(T) \cdot I_{K_1 \leq S(T) \leq K_2} \right] \end{aligned}$$

Now change measure and use the stock price as numeraire:

$$\begin{aligned} V &= \frac{S(0)}{K_1} \cdot \mathbb{E}^Q \left[\underbrace{\frac{e^{-rT} \cdot S(T)}{S(0)}}_{\frac{dQ^*}{dQ}} \cdot I_{K_1 \leq S(T) \leq K_2} \right] \\ &= \frac{S(0)}{K_1} \cdot \mathbb{E}^{Q^*} \left[I_{K_1 \leq S(T) \leq K_2} \right] \\ &= \frac{S(0)}{K_1} \cdot \left\{ Q^* \left(S(T) \geq K_1 \right) - Q^* \left(S(T) \geq K_2 \right) \right\} \end{aligned}$$

Under Q we have

$$\begin{aligned} \frac{dS}{S} &= r dt + \sigma dW_Q \\ dW_{Q^*} &= dW_Q - \frac{d\xi}{\xi} dW_Q \\ &= dW_Q - \sigma \underbrace{dW_Q dW_Q}_{dt} \end{aligned}$$

Under Q^* we have then

$$\begin{aligned}
\frac{dS}{S} &= \left(r+\sigma^2\right)dt + \sigma dW_{Q^*} \\
S(t) &= S(0)\cdot e^{\left(r+\sigma^2-\frac{1}{2}\cdot\sigma^2\right)t+\sigma W_Q} \\
\log S(t) &= \log S(0) + \left(r+\frac{1}{2}\cdot\sigma^2\right)\cdot t + \sigma W_{Q^*} \\
W_{Q^*} &\sim N[0,t]
\end{aligned}$$

So the fair value of the share is equal to

$$\begin{aligned}
V &= \frac{S(0)}{K_1}\cdot\left(Q^*\left(\log S(0) + \left(r+\frac{1}{2}\cdot\sigma^2\right)\cdot T + \sigma W_{Q^*}(T) \le \log K_2\right) - \ldots\right) \\
&= \frac{S(0)}{K_1}\cdot\left(\ldots - Q^*\left(\frac{W_{Q^*}(T)}{\sqrt{T}} \le \underbrace{\frac{\log\frac{K_1}{S(0)} - \left(r+\frac{1}{2}\cdot\sigma^2\right)\cdot T}{\sigma\cdot\sqrt{T}}}_{d(K_1)}\right)\right) \\
&= \frac{S(0)}{K_1}\cdot\left(N[dK_2] - N[dK_1]\right)
\end{aligned}$$

2. With dividend we only have to substitute r with $r-\delta$.

15.5 Example 5

Show that under the same assumptions as in problem 2 that

$$\mathbb{E}^{Q^*}\left[\frac{1}{X(T)}\middle|\mathcal{F}_t\right] = \frac{1}{X(t)}$$

where Q^* is the foreign risk-neutral measure.

Solution:

Using $S(t) = e^{-\int_0^t r\,du}$ under Q^*we get

$$\begin{aligned}
S(t)\cdot\frac{1}{X(t)}\cdot e^{-\int_0^t r^f du} &\hat{=} Q - \text{martingale} \\
&= e^{\int_0^t\left(r-r^f\right)du}\cdot\frac{1}{X(t)}
\end{aligned}$$

If $r = r^f$ this equals to $\frac{1}{X(t)}$.

15.6 Example 6

Consider the stochastic volatility model

$$\begin{aligned} \frac{dS}{S} &= rdt + \sqrt{v}dW_1 \\ dv &= \kappa\left(\theta - v\right)dt + \sigma dW_2 \end{aligned}$$

written under Q, where $d\left\langle W_1, W_2\right\rangle = 0$. Derive PDE's for the two probabilities in the call option pricing formula.

Solution:

1. Find d_1, so that $Q^*\left(P\left(T,u\right) > K\right) = N\left[d_1\right]$ where Q^* uses the u-maturity bond as numeraire.

   We know that

$$\begin{aligned} \sigma\left(t,s\right) &= \sigma \\ \frac{dP\left(t,u\right)}{P\left(t,u\right)} &= rdt - \left(\int_t^u \sigma\left(t,s\right)ds\right)dW\left(t\right) \\ &= rdt - \left(u-t\right)\cdot\sigma dW\left(t\right) \end{aligned}$$

   When we set $\frac{dQ^*}{dQ} = \xi$, we have

$$\frac{d\xi}{\xi} = -\left(u-t\right)\cdot\sigma dW\left(t\right)$$

   and

$$\begin{aligned} dW^* &= dW - \left(\frac{d\xi}{\xi}\right)dW \\ &= dW + \left(u-t\right)\cdot\sigma dt \end{aligned}$$

   defines a Q^*-Brownian motion. Substituting gives us

$$\frac{dP\left(t,u\right)}{P\left(t,u\right)} = rdt - \left(u-t\right)\cdot\sigma dW^*\left(t\right) + \left(u-t\right)^2\cdot\sigma^2 dt$$

   Also

$$dW^* = dW + \left(u-t\right)\cdot\sigma dt$$

   implies

$$\begin{aligned} W^*\left(t\right) &= W\left(t\right) + \int_0^t \left(u-s\right)\cdot\sigma ds \\ &= W\left(t\right) + \left(u\cdot t - \frac{t^2}{2}\right)\cdot\sigma \end{aligned}$$

So we get

$$r(t) = r(0) + \theta(t) + \sigma \cdot W^*(t) - \left(u \cdot t - \frac{t^2}{2}\right) \cdot \sigma^2$$

Now we set $\phi(t) = \theta(t) - \left(u \cdot t - \frac{t^2}{2}\right) \cdot \sigma^2$ and rewrite $r(t)$ as

$$r(t) = r(0) + \phi(t) + \sigma W^*(t)$$

The formula for $\frac{dP}{P}$ implies

$$\begin{aligned} P(T,u) &= P(0,u) \cdot e^{\int_0^T r(t)dt + \sigma^2 \cdot \int_0^T (u-t)^2 dt - \sigma \cdot \int_0^T (u-t) dW^*(t) - \frac{1}{2} \cdot \sigma^2 \cdot \int_0^T (u-t)^2 dt} \\ &= P(0,u) \cdot e^{\int_0^T r(t) + \frac{\sigma^2}{2} \int_0^T (u-t)^2 dt - \sigma \cdot \int_0^T (u-t) dW^*(t)} \\ &= P(0,u) \cdot e^{T \cdot r(0) + \int_0^T \phi(t)dt + \sigma \cdot \int_0^T W^*(t)dt + \frac{\sigma^2}{2} \cdot \int_0^T (u-t)^2 dt - \sigma \cdot \int_0^T (u-t) dW^*(t)} \end{aligned}$$

Apply Ito's Lemma to the function $f(t, W^*) = t \cdot W^*$and we get

$$\begin{aligned} df &= W^*(t)\, dt + t \cdot dW^*(t) \\ T \cdot W^*(T) - 0 \cdot W^*(0) &= \int_0^T df \\ &= \int_0^T W^*(t)\, dt \\ &= \int_0^T t dW^*(t) \end{aligned}$$

Hence,

$$\begin{aligned} \sigma \int_0^T (u-t)\, dW^*(t) &= \sigma \cdot u \cdot W^*(T) - \sigma \cdot \int_0^T t dW^*(t) \\ &= \sigma \cdot u \cdot W^*(T) + \sigma \cdot \int_0^T W^*(t)\, dt - \sigma \cdot T \cdot W^*(T) \\ &= \sigma \cdot (u-T) \cdot W^*(T) + \sigma \cdot \int_0^T W^*(t)\, dt \end{aligned}$$

Substituting in the formula for $P(T,u)$ gives

$$P(t,u) = P(0,u) \cdot e^{T \cdot r(0) + \int_0^T \phi(t)dt + \frac{\sigma^2}{2} \int_0^T \left(-u \cdot t + \frac{t^2}{2} + \frac{u^2}{2} - t \cdot u + \frac{t^2}{2}\right) dt}$$

Also

$$\begin{aligned} \int_0^T \phi(t)\, dt + \frac{\sigma^2}{2} \cdot \int_0^T (u-t)^2\, dt &= \int_0^T \theta(t)\, dt + \sigma^2 \cdot \int_0^T \left(-u \cdot t + \frac{t^2}{2} + \frac{u^2}{2} - t \cdot u + \frac{t^2}{2}\right) dt \\ &= \int_0^T \theta(t)\, dt + \sigma^2 \cdot \left(-\frac{u \cdot T^2}{2} + \frac{T^3}{3} + \frac{u^2 \cdot T}{2} - \frac{T^2 \cdot u}{2}\right) \\ &= \int_0^T \theta(t)\, dt - \sigma^2 \cdot \left(u \cdot T^2 - \frac{u^2 \cdot T}{2} - \frac{T^3}{3}\right) \end{aligned}$$

We have $P(T,u) > K$ iff $\log P(T,u) > \log K$:

$$\begin{aligned} \log P(0,u) + T \cdot r(0) + \int_0^T \theta(t)\,dt - \sigma^2 \cdot \left(u \cdot T^2 - \frac{u^2 \cdot T}{2} - \frac{T^3}{3} \right) - \sigma \cdot (u-T) \cdot W^*(T) &> \log K \\ \log \frac{P(0,u)}{K} + T \cdot r(0) + \int_0^T \theta(t)\,dt - \sigma^2 \cdot \left(u \cdot T^2 - \frac{u^2 \cdot T}{2} - \frac{T^3}{3} \right) &> \sigma \cdot (u - \end{aligned}$$

To convert $\sigma(u-T) \cdot W^*(T)$ to a standard normal, divide by $\sigma \cdot (u-T) \cdot \sqrt{T}$. Hence

$$d_1 = \frac{1}{\sigma \cdot (u-T) \cdot \sqrt{T}} \cdot \left\{ \log \frac{P(0,u)}{K} + T \cdot r(0) + \int_0^T \theta(t)\,dt - \sigma^2 \left(u \cdot T^2 - \frac{u^2 \cdot T}{2} - \frac{T^3}{3} \right) \right\}$$

2. Find d_2, so that $Q^{**}[P(T,u) > K] = N[d_2]$ where Q^{**} uses the T-maturity bond as numeraire.
   The dynamics of the T-maturity bond are

$$\frac{dP(t,T)}{P(t,T)} = rdt - (T-t) \cdot \sigma dW(t)$$

so

$$dW^{**} = dW + (T-t) \cdot \sigma dt$$

defines a Q^{**}-Brownian motion. This gives

$$\begin{aligned} W^{**}(t) &= W(t) + \int_0^t (T-s) \cdot \sigma ds \\ &= W(t) + \sigma \cdot \left(T \cdot t - \frac{t^2}{2} \right) \end{aligned}$$

So

$$r(t) = r(0) + \theta(t) - \sigma^2 \cdot \left(T \cdot t - \frac{t^2}{2} \right) + \sigma \cdot W^{**}(t)$$

The option is written on the u-maturity bond and

$$\begin{aligned} \frac{dP(t,u)}{P(t,u)} &= rdt - (u-t) \cdot \sigma dW(t) \\ &= rdt + \sigma^2 \cdot (T-t) \cdot (u-t)\,dt - (u-t) \cdot \sigma dW^{**}(t) \end{aligned}$$

So

$$\begin{aligned} P(T,u) &= P(0,u) \cdot e^{\int_0^T r(t)dt + \sigma^2 \cdot \int_0^T (T-t)\cdot(u-t)dt - \sigma \cdot \int_0^T (u-t)dW^{**}(t) - \frac{\sigma^2}{2} \cdot \int_0^T (u-t)^2 dt} \\ &= P(0,u) \cdot e^{T \cdot r(0) + \int_0^T \theta(t)dt - \sigma^2 \cdot \left(\frac{2 \cdot T^3}{3} - \frac{u^2 \cdot T}{2} \right) - \sigma \cdot (u-T) \cdot W^{**}(T)} \end{aligned}$$

Thus $P(T,u) > K$ iff

$$\log P(0,u) + T \cdot r(0) + \int_0^T \theta(t)\,dt - \sigma^2 \cdot \left(\frac{2 \cdot T^3}{3} - \frac{u^2 \cdot T}{2} \right) - \sigma \cdot (u-T) \cdot W^{**}(T) > \log K$$

So we define d_2 as

$$d_2 = \frac{1}{\sigma \cdot (u-T) \cdot \sqrt{T}} \cdot \left\{ \log \frac{P(0,u)}{K} + T \cdot r(0) + \int_0^T \theta(t)\,dt - \sigma^2 \cdot \left(\frac{2 \cdot T^3}{3} - \frac{u^2 \cdot T}{2} \right) \right\}$$

3. The value of the options is therefore

$$\begin{aligned} &\mathbb{E}^Q \left[e^{-\int_0^T r(t)dt} \cdot P(T,u) \cdot I_{P(T,u)>K} \right] - \mathbb{E}^Q \left[e^{-\int_0^T r(t)dt} \cdot K \cdot I_{P(T,u)>K} \right] = \\ &= P(0,u) \cdot Q^* (P(T,u) > K) - K \cdot P(0,T) \cdot Q^{**} (P(T,u) > K) \\ &= P(0,u) \cdot N[d_1] - K \cdot P(0,T) \cdot N[d_2] \end{aligned}$$